Research Article

Journal of Scientific Enquiry
Vol: 4 (1), Pages: 23 - 30

Some renormings of c_0 with the Weak Fixed Point Property

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Received: 16.06.2024; accepted: 07.08.2024; published online: May, 2025

Abstract

One of the open problems in metric fixed point theory is: Does every equivalent renorming of the classical Banach space $(c_0, \|.\|_\infty)$ have the weak fixed point property? In this paper, we present some equivalent renormings of $(c_0, \|.\|_\infty)$ with the weak fixed point property and also give an alternative proof for the weak fixed point property of $(c_0, \|.\|_\infty)$.

Keywords: Nonexpansive mappings, Weak fixed point property.

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1. Introduction

Let K be a nonempty closed bounded convex subset of a normed linear space X. A mapping $T: K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. The set K is said to have the fixed point property (fpp) if every nonexpansive self-mapping on K has a fixed point in K. We say that X has the fpp if every closed bounded convex subset of X has the fpp and also we say X has the weak fixed point property (wfpp) if every weakly compact convex subset of X has the fpp. In general, a closed bounded convex subset of a Banach space X need not have the fpp [6]. In 1965, kirk [13] showed that every Banach space with normal structure has the wfpp. It is known that every compact convex subset of a Banach space has normal structure [6]. Also, every uniformly convex Banach space has normal structure [6]. In 1976, Karlovitz [7] proved that normal structure is not an essential condition for the wfpp of a Banach space. Thenceforth, numerous authors have studied various types of geometric conditions which give rise to the wfpp for a Banach space (one can refer [1, 5, 6, 9, 14]).

"Does every Banach space have the wfpp?" was an unsolved problem in metric fixed point theory for a long time. In 1981, Alspach [3] showed that $L_1[0,1]$ does not have the wfpp. It is known that c_0 with the sup norm does not have the fpp [6]. However, in 1981 Maurey [2] proved that c_0 with the sup norm has the wfpp. Note that c_0 with the sup norm does not have normal structure [6]. It is unknown whether every equivalent renorming of $(c_0, \|.\|_\infty)$ has the wfpp or not. Another open question in metric fixed point theory is "Do all reflexive spaces have the fpp?" To know more about all these open problems one can refers [1, 4- 6, 8-9, 12, 14-15].

In this paper, we prove the following:

Let $(X, \|.\|)$be a Banach space and $\{e_n\}$ be a Schauder basis of X. Suppose $\{f_n\}$ is the associated sequence of coordinate functionals. Let $|.|$ be a norm on X such that $(X, |.|)$ be an equivalent renorming of $(X, \|.\|)$. Then $(X, |.|)$has the wfpp if there is a non negative integer Nsatisfying the following condition:

Let $x \in X$ and $|x| > a$, for some $a > 0$. Then there exist positive integers j, m and non negative integers $n_1 < n_2 < \cdots < n_m$ such that

$$|x| \geq \left|f_{j+n_1}(x)\right| + \cdots + \left|f_{j+n_m}(x)\right| > a,$$

$$|x - z| \geq \left|f_{j+n_1}(x-z)\right| + \cdots + \left|f_{j+n_m}(x-z)\right|, \forall z \in X$$

where $(n_m - n_1) \leq N$.

In addition we prove the following:

Let $(X, \|.\|)$ be a Banach space and $\{e_n\}$ be a normalized Schauder basis of X. Suppose $\{f_n\}$ is the associated sequence of coordinate functionals. Define, $|.|$ is a norm on X such that $|x| = |f_1(x)| + \cdots + |f_i(x)| + \sup_{n \geq (i+1)} |f_n(x)|$, for all $x \in X$,where $i \geq 1$ is an integer. Suppose

$(X, |.|)$ is a Banach space. Then $(X, |.|)$ does not have normal structure but has the wfpp.

From the above, we give examples of some equivalent renormings of $(c_0, \|.\|_\infty)$with the wfpp and we derive an alternate proof for the wfpp of the classical Banach space $(c_0, \|.\|_\infty)$.

2. Main results

Let C be a nonempty weakly compact convex subset of a Banach space and T be a nonexpansive map on C.

Definition 2.1. *A nonempty weakly compact convex subset K of C is said to be minimal invariant under T if K is invariant under $T$and no proper nonempty weakly compact convex subset of K is invariant under T.*

Obviously, if T has a fixed point x in C, then $\{x\}$ is a minimal invariant set under T. The following theorem ensures the existence of a minimal invariant set for a nonexpansive map which is a consequence of Zorn's lemma

Theorem 2.2. [6] *Let C be a nonempty weakly compact convex subset of a Banach space X. Let Tbe a nonexpansive map on C. Then there exists a weakly compact convex subset K of C which is a minimal invariant set under T.*

Definition 2.3. A sequence $\{x_n\}$ in C is said to be an approximate fixed point sequence (afps) of Tif $\|x_n - Tx_n\| \to 0$. The existence of such a sequence is a result of Banach contraction principle.

Theorem 2.4. [6] *Let C be a nonempty closed bounded convex subset of a Banach space X and T be a nonexpansive map on C. Then T has an afps in C.*

In general, an afps need not converge to a point in$C$. But if an afps converges to a point x in C,then x is a fixed point of T.

The next theorem is known as Goebel Karlovitz lemma [7].

Theorem 2.5. [7] *Let X be a Banach space. Let K be a nonempty weakly compact convex subset of X and $T: K \to K$ be a nonexpansive map. Suppose K is minimal invariant under T and $\{x_n\}$ is an afps of T in K. Then $\lim_{n\to\infty} \| x_n - x\| = diam(K)$, for all $x \in K$.*

The above theorem shows that the behaviour of an afps is quite unusual. But this unusual behaviour is not too bad either. Actually, this behaviour is very useful for the study of wfpp of a Banach space.

The following theorem is an important result in the direction of wfpp due to Maurey [2] .

Theorem 2.6. [2, 16] *Let K be a nonempty closed bounded convex subset of a Banach space X. Suppose $T: K \to K$ is a nonexpansive map and $\{x_n\}, \{y_n\}$ are afps of T. Then there exists an afps $\{z_n\}$ of Tsuch that*

$$\text{limsup}\|x_n - z_n\| \leq \frac{diam(K)}{2}$$

$$\text{limsup}\|y_n - z_n\| \leq \frac{diam(K)}{2}$$

In 1948, Brodskii and Milman [10] introduced the idea of normal structure to study the existence of fixed points of isometries.

Definition 2.7. [6,10] *A nonempty convex subset K of a Banach space X is said to have normal structure if for every closed bounded convex subset S of K with $diam(S) > 0$, there exists $x \in S$ such that $r_x(S) = \sup\{\|x-y\|: y \in S\} < diam(S)$.*

The following theorem yields a characterization of normal structure.

Theorem 2.8. [6] *A nonempty bounded convex subset K of a Banach space X has normal structure if and only if it does not contain a non constant sequence $\{x_n\}$ such that*

$$\lim_{n\to\infty} dist\,(x_{n+1}, co\{x_1, x_2, \dots, x_n\})$$

$$.= diam\{x_1, x_2, \dots, x_n, \dots\}.$$

Let X be a Banach space and $\{e_n\}$ be a Schauder basis of X. Let $x \in X$. Then there exists a unique sequence of scalars $\{\alpha_n\}$ such that $x = \sum \alpha_n e_n$. The support of x with respect to $\{e_n\}$ is $supp(x) = \{n \in \mathbb{N}: \alpha_n \neq 0\}$. For each $n \in \mathbb{N}$, define $f_n: X \to \mathbb{R}$ as $f_n(x) = \alpha_n, \forall x \in X$. Then $\{f_n\}$ is the associated sequence of coordinate functionals with respect to the Schauder basis $\{e_n\}$. If $\{e_n\}$ is a normalized Schauder basis that is $\|e_n\| = 1, \forall n$, then $\{f_n\}$ is uniformly bounded [11].

The next lemma is a modification of Proposition 0.14 given in [1].

Lemma 2.9. [1] *Let $(X, \|.\|)$ be a Banach space with Schauder basis $\{e_n\}$ and let N be a non negative integer. Suppose $\{x_n\}$ is a sequence in X which converges weakly to 0. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{u_k\}$ such that*

(i) $\|x_{n_k} - u_k\| \to 0$

(ii) There exist positive integers $N_0, N_1, \dots, N_k, \dots$ such that $N_{k-1} + N < N_k, k = 1,2, \dots.$ and $supp(u_k) \subset (N_{k-1}, N_k]$ where $(N_{k-1}, N_k] = \{j \in \mathbb{N}: N_{k-1} < j \leq N_k\}$.

Proof: Let $\{P_n\}$ be the sequence of natural projections with respect to Schauder basis $\{e_n\}$. Let $\{\epsilon_k\}_{k\geq 0}$ be a sequence of positive real numbers converging to 0.

Let $n_0 \in \mathbb{N}$. Then $\|P_n(x_{n_0}) - x_{n_0}\| \to 0$ as $n \to \infty$. Therefore there exists a $N_0 \in \mathbb{N}$ such that $\|P_{N_0}(x_{n_0}) - x_{n_0}\| < \epsilon_0$.

Now $\|P_{N_0}(x_n)\| \to 0$ as $n \to \infty$. So there exists a $n_1 > n_0$ such that $\|P_{N_0}(x_{n_1})\| < \epsilon_1$.

Again $\|P_n(x_{n_1}) - x_{n_1}\| \to 0$ as $n \to \infty$.

Therefore there exists a $N_1 > N_0 + N$ such that $\|P_{N_1}(x_{n_1}) - x_{n_1}\| < \epsilon_1$.

Now since $\|P_{N_1}(x_n)\| \to 0$ as $n \to \infty$. So there exists a $n_2 > n_1$ such that $\|P_{N_1}(x_{n_2})\| < \epsilon_2$.

Continuing this process we get $n_0 < n_1 < \dots < n_k < \cdots$ and $N_0 < N_1 < \dots < N_k < \cdots$ such that

$\|P_{N_k}(x_{n_k}) - x_{n_k}\| < \epsilon_k$ and $\|P_{N_{k-1}}(x_{n_k})\| < \epsilon_k$.

Let $u_k = (P_{N_k} - P_{N_{k-1}})(x_{n_k})$. Then $\|x_{n_k} - u_k\| \leq \|P_{N_k}(x_{n_k}) - x_{n_k}\| + \|P_{N_{k-1}}(x_{n_k})\| < 2\epsilon_k$.

Therefore $\|x_{n_k} - u_k\| \to 0$ and $supp(u_k) \subset (N_{k-1}, N_k], N_{k-1} + N < N_k, k = 1,2, \dots.$

Theorem 2.10. *Let $(X, \|.\|)$ be a Banach space and $\{e_n\}$ be a Schauder basis of X. Suppose $\{f_n\}$ is the associated sequence of coordinate functionals. Let $|.|$ be a norm on X such that $(X, |.|)$ be an equivalent renorming of $(X, \|.\|)$. Then $(X, |.|)$ has the wfpp if there is a non negative integer N satisfying the following condition:*

Let $x \in X$ and $|x| > a$, for some $a > 0$. Then there exist positive integers j, m and non negative integers $n_1 < n_2 < \cdots < n_m$ such that

$$|x| \geq |f_{j+n_1}(x)| + \cdots + |f_{j+n_m}(x)| > a,$$

$$|x - z| \geq |f_{j+n_1}(x-z)| + \cdots + |f_{j+n_m}(x-z)|, \forall z \in X$$

where $(n_m - n_1) \leq N$.

Proof: Suppose $(X, |.|)$ does not have the wfpp. Then there exists a nonempty weakly compact convex set K and a nonexpansive map $T: K \to K$ which has no fixed point in K. Assume that K is minimal invariant and $diam_{|.|}(K) = 1$.

Let $\{x_n\}$ be an afps of T in K.

Without loss of generality assume that $0 \in K$ and $\{x_n\}$ converges weakly to 0.

By Lemma 2.9, there is a sequence $\{u_k\}$and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $|x_{n_k} - u_k| \to 0$ and $supp(u_k) \subset (N_{k-1}, N_k]$, where $N_{k-1} + N < N_k, k = 1,2, \ldots$.

Now, $\{x_{n_k}\}$ and $\{x_{n_{k+2}}\}$ are afps of T. Then by Theorem 2.6, there exists an afps$\{z_k\}$ such that

$\text{limsup}|x_{n_k} - z_k| \le \frac{1}{2}$,$\text{limsup}|x_{n_{k+2}} - z_k| \le \frac{1}{2}$

Therefore, $\text{limsup}|u_k - z_k| \le \frac{1}{2}$,$\text{limsup}|u_{k+2} - z_k| \le \frac{1}{2}$.

Now by Goebel Karlovitz Lemma (Theorem 2.5), $|z_k - x| \to 1$, for all $x \in K$.

Let $0 < \epsilon < \frac{1}{4}$. Choose, $k_0 \in \mathbb{N}$ such that

$$|z_{k_0}| > (1-\epsilon), |u_{k_0} - z_{k_0}| < \frac{1}{2} + \epsilon,$$

$|u_{k_0+2} - z_{k_0}| < \frac{1}{2} + \epsilon$.

So, by the assumption, there exist positive integersj, m and non negative integers $n_1 < n_2 < \cdots < n_m$ such that

$$|z_{k_0}| \ge |f_{j+n_1}(z_{k_0})| + \cdots + |f_{j+n_m}(z_{k_0})| > 1 - \epsilon$$

$$|z_{k_0} - z| \ge |f_{j+n_1}(z_{k_0} - z)| + \cdots + |f_{j+n_m}(z_{k_0} - z)|,$$

for all $z \in X$, where $(n_m - n_1) \le N$.

Now, $|f_{j+n_1}(u_{k_0})| + \cdots + |f_{j+n_m}(u_{k_0})| \ge |f_{j+n_1}(z_{k_0})| + \cdots + |f_{j+n_m}(z_{k_0})|$
$-|f_{j+n_1}(z_{k_0} - u_{k_0})| + \cdots + |f_{j+n_m}(z_{k_0} - u_{k_0})|$
$.> (1-\epsilon) - |u_{k_0} - z_{k_0}| > (1-\epsilon) - \left(\frac{1}{2} + \epsilon\right) > 0$

Similarly $|f_{j+n_1}(u_{k_0+2})| + \cdots + |f_{j+n_m}(u_{k_0+2})| > 0$. Since$|f_{j+n_1}(u_{k_0})| + \cdots + |f_{j+n_m}(u_{k_0})| > 0$ there exists n_t such that$|f_{j+n_t}(u_{k_0})| > 0$.

So, $j + n_t \in (N_{k_0-1}, N_{k_0}]$.

Also since, $|f_{j+n_1}(u_{k_0+2})| + \cdots + |f_{j+n_m}(u_{k_0+2})| > 0$. There exists n_s such that $|f_{j+n_s}(u_{k_0+2})| > 0$. So, $j + n_s \in (N_{k_0+1}, N_{k_0+2}]$.

Now, $N_{k_0} + N < N_{k_0+1}$. Thus, we have $|n_t - n_s| > N$,which is a contradiction.

Corollary 2.11. *Let $(X, \|.\|)$ be a Banach space and $\{e_n\}$ be a normalized Schauder basis of X. Suppose$\{f_n\}$ is the associated sequence of coordinate functionals. Define, $|.|$ is a norm on X such that $|x| = \sup_{n\in\mathbb{N}}|f_n(x)|$, for all $x \in X$. Suppose $(X, |.|)$ is a Banach space. Then $(X, |.|)$ does not have normal structure but has the wfpp.*

Proof: Since $\{e_n\}$ is a normalized Schauder basis, there exists a $M > 0$ such that $\|f_n\| \le M, \forall n \in \mathbb{N}$. Therefore $|x| = \sup_{n\in\mathbb{N}}|f_n(x)| \le M\|x\|$. Now since $(X, |.|)$ is a Banach space, by Open mapping theorem $(X, |.|)$is an equivalent renorming of $(X, \|.\|)$.

Let $K = \overline{co}\{e_1, e_2, \cdots, e_n, \cdots\}$. Then K is a closed bounded convex set with $diam(K) > 0$ in $(X, |.|)$. Now $|e_n - e_m| = 1, m \ne n$. Therefore $diam\{e_1, e_2, \cdots, e_n, \cdots\} = 1$.

Now$\lim_{n\to\infty} dist\,(e_{n+1}, co\{e_1, e_2, \cdots, e_n\}) = diam\{e_1, \cdots, e_n, \cdots\}$.Hence by characterization of normal structure (Theorem 2.8), K does not have normal structure. Therefore $(X, |.|)$ does not have normal structure.

Now let $x \in X$ and $|x| > a$, for some $a > 0$.

Then there exists a positive integer j such that

$$|x| \ge |f_j(x)| > a,$$

$$|x - z| \ge |f_j(x - z)|, \text{for all} z \in X.$$

Now by taking $N = 0, m = 1, n_1 = 0$, in Theorem 2.10, we get the conclusion.

Example 2.12. Consider $(c_0, \|.\|_\infty)$ with the standard Schauder basis $\{e_n\}$, where $e_n = (x(1), x(2), \cdots, x(n), \cdots), x(n) = 1, x(i) = 0, i \ne n$.Suppose $\{f_n\}$ is the associated sequence of coordinate functionals. Then $|x| = \sup_{n\in\mathbb{N}}|f_n(x)| = \sup_{n\in\mathbb{N}}|x(n)| = \|x\|_\infty$,for all $x \in c_0$. Then by Corollary 2.11, $(c_0, \|.\|_\infty)$does not have the normal structure but has the wfpp.

Example 2.13. Consider $(c_0, |.|)$, where $|x| = \sup_{n\in\mathbb{N}}(|x(n)| + |x(n+1)| + \cdots + |x(n+i)|)$, for all $x \in c_0$ and $i \ge 1$ is an integer. Then $|.|$ is an equivalent norm of $\|.\|_\infty$. Let $\{e_n\}$ be the standard Schauder basis of $(c_0, \|.\|_\infty)$ and $\{f_n\}$ be the associated sequence of coordinate functionals, where $e_n = (x(1), x(2), \cdots, x(n), \cdots), x(n) = 1, x(i) = 0, i \neq n$and $f_n(x) = |x(n)|, \forall x \in c_{0.}$

Now let $x \in c_0$ and $|x| > a$, for some $a > 0$. Then there exists a positive integer j such that

$$|x| \geq |f_j(x)| + \cdots + |f_{j+i}(x)| > a,$$

$$|x - z| \geq |f_j(x - z)| + \cdots + |f_{j+i}(x - z)|, \text{for all} z \in c_{0.}$$

Now by taking $N = i, m = (i+1), n_1 = 0, n_2 = 1, \ldots, n_m = i$, in Theorem 2.10, we get $(c_0, |.|)$ has the wfpp.

Let $K = \overline{co}\{x_n : n \in \mathbb{N}\} = \overline{co}\{e_{1+(n-1)(i+1)} : n \in \mathbb{N}\}$.

Then K is a closed bounded convex set with

$diam(K) > 0$in $(c_0, |.|)$. Now $|x_n - x_m| = 1, m \neq n$.Therefore $diam\{x_1, x_2, \cdots, x_n, \cdots\} = 1$.

Now $\lim_{n\to\infty} dist\,(x_{n+1}, co\{x_1, x_2, \cdots, x_n\})$

$. = diam\{x_1, \cdots, x_n, \cdots\}$.

Hence by characterization of normal structure (Theorem 2.8), K does not have normal structure.

Theorem 2.14. *Let $(X, \|.\|)$ be a Banach space and $\{e_n\}$ be a normalized Schauder basis of X.Suppose $\{f_n\}$ is the associated sequence of coordinate functionals. Define, $|.|$ is a norm on Xsuch that $|x| = |f_1(x)| + \cdots + |f_i(x)| + \sup_{n\geq(i+1)}|f_n(x)|$, for all $x \in X$, where $i \geq 1$ is an integer. Suppose $(X, |.|)$ is a Banach space. Then $(X, |.|)$ does not have normal structure but has the wfpp.*

Proof: Since $\{e_n\}$ is a normalized Schauder basis, there exists a $M > 0$ such that $\|f_n\| \leq M, \forall n \in \mathbb{N}$.Therefore $|x| = |f_1(x)| + \cdots + |f_i(x)| + \sup_{n\geq(i+1)}|f_n(x)|$

$. \leq (i+1)M\|x\|, \forall x \in X$.

Now since $(X, |.|)$ is a Banach space, by Open mapping theorem $(X, |.|)$ is an equivalent renorming of $(X, \|.\|)$.

Let $K = \overline{co}\{x_1, x_2, \cdots, x_n, \cdots\} = \overline{co}\{e_{i+1}, e_{i+2}, \cdots, e_{i+n}, \cdots\}$.Then K is a closed bounded convex set with $diam(K) > 0$ in $(X, |.|)$. Now $|x_n - x_m| = 1, m \neq n$.Therefore $diam\{x_1, x_2, \cdots, x_n, \cdots\} = 1$.

Now $\lim_{n\to\infty} dist\,(x_{n+1}, co\{x_1, x_2, \cdots, x_n\})$

$. = diam\{x_1, x_2, \cdots, x_n, \cdots\}$.

Hence by characterization of normal structure (Theorem 2.8), K does not have normal structure.

Suppose $(X, |.|)$ does not have the wfpp. Then there exists a nonempty weakly compact convex set $K$and a nonexpansive map $T: K \to K$ which has no fixed point in K. Assume that K is minimal invariant and $diam_{|.|}(K) = 1$.

Let $\{x_n\}$ be an afps of T in K. Without loss of generality assume that $0 \in K$and$\{x_n\}$ converges weakly to 0.

By Lemma 2.9, there is a sequence $\{u_k\}$and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $|x_{n_k} - u_k| \to 0$and $supp(u_k) \subset (N_{k-1}, N_k]$, where $N_{k-1} + i < N_k, k = 1,2, \ldots$.

Now, $\{x_{n_k}\}$ and $\{x_{n_{k+2}}\}$ are afps of T. Then by Theorem 2.6, there exists an afps $\{z_k\}$such that

$\limsup|x_{n_k} - z_k| \leq \frac{1}{2}$,$\limsup|x_{n_{k+2}} - z_k| \leq \frac{1}{2}$

Therefore,

$\limsup|u_k - z_k| \leq \frac{1}{2}$,$\limsup|u_{k+2} - z_k| \leq \frac{1}{2}$

Now by Goebel Karlovitz Lemma (Theorem 2.5), $|z_k - x| \to 1$, for all $x \in K$.

Let $0 < \epsilon < \frac{1}{4}$. Choose, $k_0 \in \mathbb{N}(k_0 > 1)$ such that

$$|z_{k_0}| > (1-\epsilon), |u_{k_0} - z_{k_0}| < \frac{1}{2} + \epsilon, |u_{k_0+2} - z_{k_0}| < \frac{1}{2} + \epsilon$$

So, there exists a positive integer $j \geq (i+1)$ such that $|z_{k_0}| \geq |f_1(z_{k_0})| + \cdots + |f_i(z_{k_0})| + |f_j(z_{k_0})| > (1-\epsilon)$,

$$|z_{k_0} - z| \geq |f_1(z_{k_0} - z)| + \cdots + |f_i(z_{k_0} - z)| + |f_j(z_{k_0} - z)|,$$

for all $z \in X$.

Now, $|f_1(u_{k_0})| + \cdots + |f_i(u_{k_0})| + |f_j(u_{k_0})|$

$.\geq |f_1(z_{k_0})| + \cdots + |f_i(z_{k_0})| + |f_j(z_{k_0})|$
$-|f_1(z_{k_0} - u_{k_0})| + \cdots + |f_i(z_{k_0} - u_{k_0})| +$
$|f_j(z_{k_0} - u_{k_0})|$

$$.> (1-\epsilon) - |u_{k_0} - z_{k_0}| > (1-\epsilon) - \left(\frac{1}{2} + \epsilon\right) > 0.$$

Similarly, $|f_1(u_{k_0+2})| + \cdots + |f_i(u_{k_0+2})| + |f_j(u_{k_0+2})| > 0$.

Since $|f_1(u_{k_0})| + \cdots + |f_i(u_{k_0})| + |f_j(u_{k_0})| > 0$, and $supp(u_{k_0}) \subset (N_{k_0-1}, N_{k_0}], N_{k_0-1} > i$, we have $|f_j(u_{k_0})| > 0$. Also $|f_1(u_{k_0+2})| + \cdots + |f_i(u_{k_0+2})| + |f_j(u_{k_0+2})| > 0$, implies $|f_j(u_{k_0+2})| > 0$.

This is a contradiction.

Example 2.15. Consider $(c_0, \|.\|_\infty)$ with the standard Schauder basis $\{e_n\}$, where $e_n = (x(1), x(2), \cdots, x(n), \cdots), x(n) = 1, x(i) = 0, i \neq n$.Suppose $\{f_n\}$ is the associated sequence of coordinate functionals. Define, $|.|$ is a norm on c_0 such that $|x| = |x(1)| + \cdots + |x(i)| + \sup_{n \geq (i+1)} |x(n)|$, for all $x \in c_0$,where $i \geq 1$ is an integer. Then $|.|$ is an equivalent norm of $\|.\|_\infty$. Therefore $(c_0, |.|)$ is a Banach space. Now

$|x| = |x(1)| + \cdots + |x(i)| + \sup_{n \geq (i+1)} |x(n)|$

$.= |f_1(x)| + \cdots + |f_i(x)| + \sup_{n \geq (i+1)} |f_n(x)|$,

for all $x \in c_0$.

Hence by Theorem 2.14, $(c_0, |.|)$does not have normal structure but has the wfpp.

Example 2.16. Consider $(c_0, \|.\|_\infty)$ and let $\{e_n\}$ be the standard Schauder basis, where $e_n = (x(1), x(2), \cdots), x(n) = 1, x(i) = 0, i \neq n$. Define, $|.|$ is a norm on c_0 such that $|x| = max\{a\sup_{n \in \mathbb{N}} |x(n) + x(n+1)|, b\|x\|_\infty\}$ for all $x \in c_0$, where $a > 0, b > 0$.

Let $K = \overline{co}\{x_1, x_2, \cdots, x_n, \cdots\} = \overline{co}\{e_2, e_4, \cdots, e_{2n}, \cdots\}$.Then K is a closed bounded convex set with $diam(K) > 0$ in $(c_0, |.|)$. Now $|x_n - x_m| = max\{a, b\}, m \neq n$. Therefore $diam\{x_1, x_2, \cdots, x_n, \cdots\} = max\{a, b\}$.

Now $\lim_{n\to\infty} dist\,(x_{n+1}, co\{x_1, x_2, \cdots, x_n\})$

$$.= diam\{x_1, x_2, \cdots, x_n, \cdots\}.$$

Hence by characterization of normal structure (Theorem 2.8), K does not have normal structure.

Suppose $(c_0, |.|)$ does not have the wfpp. Then there exists a nonempty weakly compact convex set $K$and a nonexpansive map $T: K \to K$ which has no fixed point in K. Assume that K is minimal invariant and $diam_{|.|}(K) = 1$.

Let $\{x_n\}$ be an afps of T in K.Without loss of generality assume that $0 \in K$ and $\{x_n\}$converges weakly to 0.

By Lemma 2.9, there is a sequence $\{u_k\}$and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $|x_{n_k} - u_k| \to 0$and $supp(u_k) \subset (N_{k-1}, N_k]$ where $N_{k-1} + 1 < N_k, k = 1,2, \ldots$.

Now, $\{x_{n_k}\}$ and $\{x_{n_{k+2}}\}$ are afps of T. Then by Theorem 2.6, there exists an afps $\{z_k\}$ such that

$\limsup |x_{n_k} - z_k| \leq \frac{1}{2} \limsup |x_{n_{k+2}} - z_k| \leq \frac{1}{2}$,

Therefore,

$\limsup |u_k - z_k| \leq \frac{1}{2}, \limsup |u_{k+2} - z_k| \leq \frac{1}{2}$

Now by Goebel Karlovitz Lemma (Theorem 2.5), $|z_k - x| \to 1$, for all $x \in K$.

Let $0 < \epsilon < \frac{1}{4}$. Choose, $k_0 \in \mathbb{N}$ such that

$|z_{k_0}| > (1-\epsilon), |u_{k_0} - z_{k_0}| < \frac{1}{2} + \epsilon, |u_{k_0+2} - z_{k_0}| < \frac{1}{2} + \epsilon$.

Now $|z_{k_0}| = max\{a\sup_{n \in \mathbb{N}} |z_{k_0}(n) + z_{k_0}(n+1)|, b\|z_{k_0}\|_\infty\}$

Suppose $a\sup_{n \in \mathbb{N}} |z_{k_0}(n) + z_{k_0}(n+1)| > 1 - \epsilon$. Then there exists a j such that

$$a|z_{k_0}(j) + z_{k_0}(j+1)| > 1 - \epsilon.$$

Now $a|u_{k_0}(j) + u_{k_0}(j+1)| \geq a|z_{k_0}(j) + z_{k_0}(j+1)| - a|(z_{k_0} - u_{k_0})(j) + (z_{k_0} - u_{k_0})(j+1)| > (1-\epsilon) - |u_{k_0} - z_{k_0}| > (1-\epsilon) - \left(\frac{1}{2} + \epsilon\right) > 0$

So $\left|u_{k_0}(j) + u_{k_0}(j+1)\right| > 0$.

Similarly $\left|u_{k_0+2}(j) + u_{k_0+2}(j+1)\right| > 0$.

This is a contradiction since $N_{k_0} + 1 < N_{k_0+1}$.

In a similar way, we can get a contradiction if $\left\|z_{k_0}\right\|_\infty > (1-\epsilon)$.

Remark 2.17. Consider $(l^p, |.|), 1 \le p < \infty$ (where $|.|$ as in Example 2.12 or 2.13 or 2.15 or 2.16) and let K be a weakly compact convex subset of $(l^p, |.|)$. Suppose $T: K \to K$ is a nonexpansive mapping. Then K is a weakly compact convex subset $(c_0, |.|)$. Since $(c_0, |.|)$ has the wfpp, T has a fixed point in K. Hence $(l^p, |.|), 1 \le p < \infty$ has the wfpp. Note that $(l^p, |.|), 1 \le p < \infty$ is not a Banach space.

Remark 2.18. We don't know in Theorem 2.8, if the assumption "$(X, |.|)$ is an equivalent renorming of $(X, \|.\|)$" can be dropped or not. So, the following question arises.

Question: Does $(C[a,b], |.|)$ have the wfpp?, where$|x| = \sup_{n\in\mathbb{N}} |f_n(x)|, for all x \in C[a,b]$and $\{f_n\}$is the associated sequence of coordinate functionals concerning a Schauder basis $\{e_n\}$ of $(C[a,b], \|.\|_\infty)$.Note that $(C[a,b], \|.\|_\infty)$ does not have the wfpp, since every separable Banach space is isometrically isomorphic to a subspace of $(C[a,b], \|.\|_\infty)$.

Acknowledgments: The author thank the referee for valuable comments and suggestions.

References

[1] A. G. Aksoy, M. A. Khamsi, Nonstandard methods in fixed point theory, Universitext, Springer-Verlag, New York (1990).

[2] B. Maurey, *Points fixes des contractions sur un convexe ferme deL^1*, Seminaire danalyse fonctionelle, Ecole Polytechnique, Palaiseau, Expose No. VIII,(1981).

[3] D. E. Alspach, *A fixed point free nonexpansive Map,* Proc. Amer. Math. Soc. 82, 423-424(1981).

[4] E. Llorens-Fuster, *Some open problems in metric fixed point theory,* https://grupo.us.es/gfqm127/vworkshop/charlas/e_Llorens.pdf

[5] J. M. Ayerbe Toledano, T. Domnguez Benavides, G. Lopez Acedo, *Measures of noncompactness in metric fixed point theory*, Vol. 99 of Operator Theory: Advances and Applications, Birkhauser Verlag, Basel (1997).

[6] K. Goebel, W. A. Kirk, *Topics in metric fixed point theory,* Vol. 28 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (1990).

[7] L. A. Karlovitz, *Existence of fixed points of nonexpansive mappings in a space without normal structure,* Pacific J. Math. 66 (1) 153-159 (1976).

[8] M.A. Japon, *Fixed point property in Banach spaces and some connections with renorming theory,* https://www.uv.es/~xiiiencmuva/charlas/xiiiMuVa_Japon.pdf

[9] M. A. Khamsi and W. A. Kirk, *An introduction to metric spaces and fixed point theory,* Pure and Applied Mathematics (New York). Wiley-Interscience, New York (2001).

[10] M. S. Brodskii and D. P. Mil'man, *On the center of a convex set,* Doklady Akad. Nauk SSSR (N.S.) 59, 837-840 (1948).

[11] R. E. Megginson, *An introduction to Banach space theory,* Graduate Texts in Mathematics, vol. 183, Springer-Verlag, New York (1998).

[12] T. Domnguez Benavides, *Some questions in metric fixed point theory,* by A.W. Kirk, revisited, Arab J. Math. 1: 431-438 (2012).

[13] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances,* Amer. Math. Monthly 72, 1004-1006 (1965).

[14] W. A. Kirk and B. Sims, *Handbook of metric fixed point thoery,* Kluwer Academic Publishers, Printed in the Netherlands (2004).

[15] W. A. Kirk, *Some questions in metric fixed point theory,* In: Recent Advances on Metric Fixed Point Theory (Seville, 1995), pp. 73-97, Ciencias, 48, Univ. Sevilla, Sevilla (1996).

[16] Y. Benyamini, J. Lindenstrauss, *Geometric nonlinear functional analysis.* Vol. 1, Vol. 48 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI (2000).