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Some results on gradient almost quasi-Yamabe and gradient Yamabe solitons

Jhantu Das[1], Anirban Mandal[1], Kalyan Halder[1*], Arindam Bhattacharyya[2]

[1] Department of Mathematics, Sidho-Kanho-Birsha University, Purulia, West Bengal-723104, India

[2] Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India

*Corresponding author

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Abstract

In this current article, we study the gradient almost quasi-Yamabe solitons and gradient Yamabe solitons inside the setting of 3-dimensional Lorentzian β-Kenmotsu manifolds. It is shown that if a 3-dimensional Lorentzian β-Kenmotsu manifold admits a gradient almost quasi-Yamabe soliton, then either the manifold is of constant sectional curvature $\kappa = -\beta^2$ or the gradient of the potential function f is pointwise collinear with ξ. Also, if the second order metric tensor field g of a 3-dimensional Lorentzian β-Kenmotsu manifold is a gradient quasi-Yamabe soliton, then either the manifold is of constant sectional curvature $\kappa = -\beta^2$ or the gradient of the potential function f is pointwise collinear with ξ. We also study the existence of gradient Yamabe solitons on 3-dimensional Lorentzian β-Kenmotsu manifolds.

Keywords: Gradient Yamabe solitons, Gradient almost quasi-Yamabe solitons, Symmetric Ricci tensor of type $(0,2)$, Lorentzian β-Kenmotsu manifolds.

Emails:
dasjhantu54@gmail.com,
25anirbanmandal1994@skbu.ac.in,
drkalyanhalder@gmail.com,
bhattachar1968@yahoo.co.in

1. Introduction

The Yamabe flow was proposed by Richard S. Hamilton (for details see [7]) to construct Yamabe metrices on compact Riemannian manifolds. According to the author, a Riemannian metric g of a complete Riemannian manifold M of dimension greater than or equal to three is said to be Yamabe soliton if it satisfies

$$\frac{1}{2}\mathcal{L}_V g = (r - \lambda)g, \qquad (1.1)$$

where $\mathcal{L}_V$ denotes the Lie-derivative along the direction of V, which is called the potential vector field of the soliton, r is the scalar curvature of the manifold and λ is a constant, which is called the soliton constant. The Yamabe soliton is a self-similar solution of Yamabe flow on a Riemannian manifold (M, g) defined as follows

$$\frac{\partial}{\partial t}(g(t)) = -r(g(t)),$$

where r is the scalar curvature of M. When establishing the characteristics of the Yamabe soliton, the potential vector field V and the soliton constant λ are crucial factors. According to whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, the Yamabe soliton is said to be shrinking, steady or expanding.

By simply generalizing the classical Yamabe soliton (1.1) and treating the soliton constant λ to a smooth function on M, in 2013 E. Barbosa and E. Ribeiro [2] introduced almost Yamabe soliton. Within the context of hypersurfaces in Euclidean spaces, an almost Yamabe soliton was completely classified by T. Seko and S. Maeta [12].

Moreover, A Yamabe soliton reduces to gradient Yamabe soliton if $V = grad(f)$, for some smooth function f defined on M. In this case, from (1.1) we have

$$\nabla^2 f = (r-\lambda)g, \qquad (1.2)$$

where $\nabla^2 f$ is the Hessian of the function f.

As a generalization of gradient Yamabe soliton, G. Huang and H. Li [9] introduced the notion of quasi-Yamabe gradient soliton. A Riemannian metric g defined on a smooth manifold M of dimension greater than two is said to be a quasi-Yamabe gradient soliton if it satisfies

$$\nabla^2 f = (r-\lambda)g + \frac{1}{p} df \otimes df \qquad (1.3)$$

for some constant λ and a positive constant p. The generalized version of quasi-Yamabe gradient soliton, so called almost quasi-Yamabe gradient soliton was initiated in [10] by setting the soliton constant λ to be a smooth function on M. In [10], the authors got some fascinating formula and give a necessary and sufficient condition under which an arbitrary compact almost Yamabe soliton is necessarily gradient. Recently, Xiaomin Chen [4] defined the notion of almost quasi-Yamabe solitons, namely the Riemannian metric g satisfies

$$\frac{1}{2}\mathcal{L}_V g = (r-\lambda)g + \frac{1}{p} V^b \otimes V^b, \qquad (1.4)$$

where V^b is the 1-form associated to V, λ is a smooth function defined on the manifold and p is a positive constant. This notion is denoted by (g, V, p, λ). The soliton (g, V, p, λ) is said to be closed if the 1-form V^b is closed. Moreover, if the potential vector field V is vanishes identically in (1.4), then the soliton (g, V, p, λ) becomes trivial. Otherwise, it will be called non-trivial. Furthermore, the soliton (g, V, p, λ) reduces to an almost Yamabe soliton when $p \longrightarrow \infty$. Moreover, if V is the gradient of a smooth function f defined on M, then the soliton (1.4) will be called almost quasi-Yamabe gradient soliton, denoted by $(g, \nabla f, p, \lambda)$. Later in [6], S. Ghosh et al. studied almost quasi-Yamabe and gradient almost quasi-Yamabe solitons in the context of Kenmotsu manifolds.

Motivated by the above studies, the present manuscript we make the contribution to investigate almost quasi-Yamabe soliton metric on 3-dimensional Lorentzian β-Kenmotsu manifolds. In this present paper, we are going to study almost quasi-Yamabe gradient solitons and gradient Yamabe solitons in the context of 3-dimensional Lorentzian β-Kenmotsu manifolds. In order to prove some results, we need to recall some definitions and some lemmas on Lorentzian β-Kenmotsu manifold admitting almost quasi-Yamabe gradient solitons, which are contained in Section 2. Next, we study 3-dimensional Lorentzian β-Kenmotsu manifolds whose metric tensor satisfies gradient almost quasi-Yamabe solitons. Finally, in the last section, we have to discuss the existence of gradient Yamabe solitons on a 3-dimensional Lorentzian β-Kenmotsu manifolds and then to establish some results on 3-dimensional Lorentzian β-Kenmotsu manifolds admitting gradient Yamabe solitons.

2. Some preliminaries on Lorentzian β-Kenmotsu manifolds

A smooth manifold M of dimensional m is called Lorentzian Kenmotsu manifold if it admits a $(1,1)$-tensor field φ, a covariant vector field ξ, a 1-form η and Lorentzian metric g which satisfy on M, respectively such that [3]

$$\eta(\xi) = -1, \tag{2.1}$$

$$\varphi^2(X) = X + \eta(X)\xi\ , \tag{2.2}$$

$$\varphi(\xi) = 0,\ \ \eta(\varphi X) = 0, \tag{2.3}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.4}$$

$$g(X,\xi) = \eta(X), \tag{2.5}$$

$$g(\varphi X, Y) = g(X, \varphi Y), \tag{2.6}$$

for any smooth vector fields X and Y on M.

Now a Lorentzian Kenmotsu manifold M is called Lorentzian β-Kenmotsu manifold if it satisfies

$$(\nabla_X\varphi)Y = \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\} \tag{2.7}$$

for any smooth vector fields X and Y on M, where ∇ denotes the Levi-Civita connection of the Lorentzian metric g and β is a constant function defined on M. Then from (2.7), it follows that

$$\nabla_X\xi = \beta\{X - \eta(X)\xi\}, \tag{2.8}$$

which gives

$$(\nabla_X\eta)Y = \beta\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{2.9}$$

for any smooth vector fields X and Y on M.

Further, on a Lorentzian β-Kenmotsu manifold M of dimension m, the following relations are satisfied for arbitrary smooth vector fields X, Y on M (*see* [1], [11]):

$$R(X,Y)\xi = \beta^2\{\eta(X)Y - \eta(Y)X\}, \tag{2.10}$$

$$R(\xi,X)Y = \beta^2\{\eta(Y)X - g(X,Y)\xi\}, \tag{2.11}$$

$$S(X,\xi) = -(m-1)\beta^2\eta(X), \tag{2.12}$$

$$Q\xi = -(m-1)\beta^2\xi, \tag{2.13}$$

where R, S and Q are the Riemann curvature tensor, the Ricci tensor of $(0,2)$, and the Ricci operator of M, respectively, and Q is given by $S(X,Y) = g(QX,Y)$.

If $\beta = 1$, the Lorentzian β-Kenmotsu manifold manifold reduces to a Lorentzian Kenmotsu manifold introduced by Mihai and et. al [17]. Also, Lorentzian β-Kenmotsu manifolds have been studied by several geometer- (*see* [1], [5], [8], [11], [13]) and many others.

It is well admitted that the Riemann curvature tensor for a 3-dimensional Riemannian manifold always satisfies

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY \\ &\quad + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}\{g(Y,Z)X \\ &\quad - g(X,Z)Y\}. \end{aligned} \tag{2.14}$$

Let M be a 3-dimensional Lorentzian β-Kenmotsu manifold. Then replacing Z by ξ in (2.14) and using the equations (2.5), (2.10) and (2.12), we infer

$$(\beta^2+\frac{r}{2})\{\eta(Y)X-\eta(X)Y\} = \{\eta(Y)QX-\eta(X)QY\}. \qquad (2.15)$$

Again, replacing Y by ξ in equation (2.15) and then using (2.1), (2.13), we obtain

$$QX=\left(\beta^2+\frac{r}{2}\right)X+\left(3\beta^2+\frac{r}{2}\right)\eta(X)\xi \qquad (2.16)$$

and hence

$$S(X,Y)=\left(\beta^2+\frac{r}{2}\right)g(X,Y) +\left(3\beta^2+\frac{r}{2}\right)\eta(X)\eta(Y). \qquad (2.17)$$

Thus, we can say that a 3-dimensional Lorentzian β-Kenmotsu manifold M is a η-Einstein manifold.

To proceed our prime theorems, we need the following Lemmas:

Lemma 2.1. *In a 3-dimensional Lorentzian β-Kenmotsu manifold M, the Riemann curvature R can be expressed as*

$$R(X,Y)Z=(2\beta^2+\frac{r}{2})\{g(Y,Z)X-g(X,Z)Y\} +(3\beta^2+\frac{r}{2})\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\} +(3\beta^2+\frac{r}{2})\{\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\} \qquad (2.18)$$

for any smooth vector fields X, Y and Z on M.

Proof: In view of (2.16) and (2.17) from equation (2.14) we obtain (2.18). This completes the proof.

Lemma 2.2. *On a 3-dimensional Lorentzian β-Kenmotsu manifold M, we have*

$$\xi(r)=-6\beta(r+6\beta^2). \qquad (2.19)$$

Proof: Taking covariant differentiation of (2.16) along an arbitrary smooth vector field Y on M and then using (2.8) we obtain

$$(\nabla_Y Q)X=\frac{1}{2}Y(r)X+\frac{1}{2}Y(r)\eta(X)\xi+ \left(3\beta^3+\frac{r\beta}{2}\right)\{g(X,Y)\xi-2\eta(X)\eta(Y)\xi+\eta(X)Y\}, \qquad (2.20)$$

for any smooth vector fields X and Y on M. Contracting the forgoing equation over Y and using the well-known formula

$trace\{Y\longrightarrow(\nabla_Y Q)X\}=\frac{1}{2}\nabla_Y(r)$,

we get

$$\xi(r)\eta(X)=-6\beta(r+6\beta^2)\eta(X).$$

Since the above relation holds for all smooth vector field X on M, we obtain

$$\xi(r)=-6\beta(r+6\beta^2).$$

This completes the proof.

Lemma 2.3 ([4]). *For an almost quasi-Yamabe gradient soliton $(g,\nabla f,p,\lambda)$, the Riemann curvature R satisfies*

$$R(X,Y)\nabla f=\frac{r-\lambda}{p}\{Y(f)X-X(f)Y\}$$

$$+X(r-\lambda)Y - Y(r-\lambda)X \qquad (2.21)$$

for any smooth vector fields X, Y *on the manifold.*

3. Gradient almost quasi-Yamabe solitons on3-Lorentzian β-Kenmotsu manifolds

In In this section, we investigate the properties of 3-dimensional Lorentzian β-Kenmotsu manifolds admitting gradient almost quasi-Yamabe solitons.

Theorem 3.1. *If a* 3*-dimensional Lorentzian* β*-Kenmotsu manifold* M*admits an almost quasi-Yamabe gradient soliton* $(g, \nabla f, p, \lambda)$, *then either the manifold* M *is of constant sectional curvature* $\kappa = -\beta^2$ *or the gradient of the potential function* f *is pointwise collinear with the characteristic vector field* ξ.

Proof: Let us assume that $(g, \nabla f, p, \lambda)$ be an almost quasi-Yamabe gradient soliton 3-dimensional Lorentzian β-Kenmotsu manifold M. Then the gradient soliton equation (1.3) we can write

$$\nabla_X \nabla f = (r-\lambda)X + \frac{1}{p}X(f)\nabla f \qquad (3.1)$$

for any smooth vector fields$X$ on M.

Now executing the inner product of (2.21) with ξ yields

$$g(R(X,Y)\nabla f, \xi) = \{\, X(r-\lambda) - \frac{r-\lambda}{p}X(f)\}\eta(Y)$$
$$-\{\, Y(r-\lambda) - \frac{r-\lambda}{p}Y(f)\,\}\eta(X). \qquad (3.2)$$

By virtue of the relation $g(R(X,Y)\nabla f, \xi) + g(R(X,Y)\xi, \nabla f) = 0$ from (2.10), we acquire

$$g(R(X,Y)\nabla f, \xi) = \beta^2\{\, \eta(Y)X(f) - \eta(X)Y(f)\}. \qquad (3.3)$$

Therefore, from (3.2) and (3.3) we have

$$\left\{X(r-\lambda) - \frac{r-\lambda}{p}X(f)\right\}\eta(Y) - \left\{Y(r-\lambda) - \frac{r-\lambda}{p}Y(f)\right\}\eta(X) = \beta^2\{\,\eta(Y)X(f) - \eta(X)Y(f)\}.$$

Replacing φX and ξ instead of the vector fields X and Y, respectively in the forgoing equation and using the equation (2.1) we get

$$\left(\beta^2 + \frac{r-\lambda}{p}\right)(\varphi X)f = (\varphi X)(r-\lambda). \qquad (3.4)$$

On the other hand, contracting (2.21) over Y and then using (2.17), we get

$$\left\{\beta^2 + \frac{r}{2} - \frac{2(r-\lambda)}{p}\right\}(\varphi X)f = -2(\varphi X)(r-\lambda). \qquad (3.5)$$

In view of (3.4), equation (3.5) reduces to

$$\left(3\beta^2 + \frac{r}{2}\right)(\varphi X)f = 0. \qquad (3.6)$$

This means that either $r = -6\beta^2$ or$(\varphi X)f = 0$.

Case (i): If $r = -6\beta^2$, then (2.17) reduces to

$$S(X,Y) = \left(\beta^2 + \frac{r}{2}\right)g(X,Y)$$

for any smooth vector fields X and Y on M. This shows that the manifold M is an Einstein manifold. Also, from (2.18) we have

$$R(X,Y)Z = -\beta^2\{g(Y,Z)X - g(X,Z)Y\}$$

for any smooth vector fields X, Y and Z on M.

This means that the manifold M is of constant sectional curvature $\kappa = -\beta^2$.

Case (ii): If $(\varphi X)f = 0$, then $(\varphi^2 X)f = 0$ and hence $\nabla f = -\xi(f)\xi$. Therefore, the gradient of the potential function f is pointwise collinear with ξ. This completes the proof.

Theorem 3.2. *If a 3-dimensional Lorentzian β-Kenmotsu manifold M admits a quasi-Yamabe gradient soliton $(g, \nabla f, p, \lambda)$, then either the manifold M is of constant sectional curvature $\kappa = -\beta^2$ or the gradient of the potential functionf of the soliton is pointwise collinear with the characteristic vector field ξ.*

Proof: Let g be a quasi-Yamabe soliton on a Lorentzian β-Kenmotsu manifold M of dimension three. Then we have from (2.21) that

$$R(X,Y)\nabla f = \frac{r-\lambda}{p}\{\,Y(f)X - X(f)Y\,\} + X(r)Y - Y(r)X \qquad (3.7)$$

Since λ is constant, equations (3.4) and (3.5) are respectively simplified as

$$\left(\beta^2 + \frac{r-\lambda}{p}\right)(\varphi X)f = (\varphi X)r, \qquad (3.8)$$

$$\left\{\beta^2 + \frac{r}{2} - \frac{2(r-\lambda)}{p}\right\}(\varphi X)f = -2(\varphi X)r. \qquad (3.9)$$

By combining (3.8) and (3.9), one immediately has

$$\left(3\beta^2 + \frac{r}{2}\right)(\varphi X)f = 0. \qquad (3.10)$$

Then maintaining the same argument as in the proof of Theorem 3.1 we can easily see that either the manifold M is of constant sectional curvature $\kappa = -\beta^2$ or the gradient of the potential function f of the soliton is pointwise collinear with the characteristic vector field ξ. This completes the proof.

Again, if $p \longrightarrow \infty$, then the soliton (1.3) reduces to an almost gradient Yamabe soliton $(g, \nabla f, \lambda)$. Thus we can state the following:

Corollary 3.3. *If a 3-dimensional Lorentzian β-Kenmotsu manifold M admits an almost gradient Yamabe soliton $(g, \nabla f, \lambda)$, then either M is of constant sectional curvature $\kappa = -\beta^2$ or the gradient of the potential function f of the soliton is pointwise collinear with the characteristic vector field ξ.*

4. Gradient Yamabe solitons on 3-Lorentz-ian β-Kenmotsu manifolds

In this section, we study the existence of gradient Yamabe solitons on a 3-Lorentzian β-Kenmotsu manifold M.

Theorem 4.1. *There does not exist a non-trivial gradient Yamabe soliton on a 3-dimensional Lorentzian β-Kenmotsu manifold M.*

Proof: If $(g, \nabla f, \lambda)$ is a gradient Yamabe soliton on M, then (1.2) takes the form:

$$\nabla_Y \nabla f = (r-\lambda)Y. \qquad (4.1)$$

Differentiating (4.1) covariantly along arbitrary smooth vector field X on M we get

$$\nabla_X \nabla_Y \nabla f = X(r)Y + (r-\lambda)\nabla_X Y. \qquad (4.2)$$

Interchanging the vector fields X and Y in equation (4.2) we obtain

$$\nabla_{\Upsilon}\nabla_{X}\nabla f=\Upsilon(r)X+(r-\lambda)\nabla_{\Upsilon}X. \quad (4.3)$$

Replacing Υ by $[X,\Upsilon]$ in equation (4.1) gives

$$\nabla_{[X,\Upsilon]}\nabla f=(r-\lambda)(\nabla_{X}\Upsilon-\nabla_{\Upsilon}X). \quad (4.4)$$

Therefore, from (4.2), (4.3) and (4.4), we obtain that

$$R(X,\Upsilon)\nabla f=X(r)\Upsilon-\Upsilon(r)X. \quad (4.5)$$

Contraction of (4.5) along Υ yields

$$S(X,\nabla f)=-2X(r), \quad (4.6)$$

which in view of the equation (2.17), yields

$$\left(\beta^{2}+\frac{r}{2}\right)X(f)+\left(3\beta^{2}+\frac{r}{2}\right)\eta(X)\xi(f)$$

$$=-2X(r). \quad (4.7)$$

Replacing ξ instead of X in (4.7) and using (2.1) and ***lemma 2.2.***, we get

$$\beta\xi(f)=-6(r+6\beta^{2}). \quad (4.8)$$

Next replacing Υ by ξ in (4.5) and using (2.11) and (2.19), we obtain

$$\beta^{2}\{X(f)\xi-\xi(f)X\}$$

$$=X(r)\xi+6\beta(r+6\beta^{2})X. \quad (4.9)$$

In view of (4.8), equation (4.9) reduces to

$$\beta^{2}X(f)=X(r). \quad (4.10)$$

Combining (4.7) and (4.10), we acquire

$$(r+6\beta^{2})\{X(f)+\eta(X)\xi(f)\}=0. \quad (4.11)$$

This implies that either

$$r=-6\beta^{2} \text{ or } X(f)+\eta(X)\xi(f)=0.$$

If possible, let $r\neq-6\beta^{2}$, then equation (4.11) gives $X(f)+\eta(X)\xi(f)=0$ and hence $\nabla(f)=-\xi(f)\xi$. Using this in (4.8) infers that

$$\beta\nabla(f)=6(r+6\beta^{2})\xi. \quad (4.12)$$

Differentiating it along Υ and then using (4.1) and taking reference of (2.1) and (2.8), we obtain

$$6\beta(r+6\beta^{2})\eta(\Upsilon)\xi-6\Upsilon(r)\xi$$

$$=\beta(5r+\lambda+36\beta^{2})\Upsilon. \quad (4.13)$$

Taking $\Upsilon=\xi$ in (4.13) and using (2.1), (2.19), we get

$$25r=\lambda-144\beta^{2}.$$

This means that the manifold M admitting gradient Yamabe soliton possesses the constant scalar curvature. Therefore, from (4.11) we have

$f = constant$.

Hence the gradient Yamabe soliton $(g,\nabla f,\lambda)$ on a 3-Lorentzian β-Kenmotsu manifold M is trivial. This completes the proof.

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