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Stability of a fractional order harvested prey–predator model in the existence of infection in prey

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The growing relationship between prey and their predator is one of the important aspects in the field of ecology and mathematical biology. On the other hand, the utility of fractional calculus in different types of mathematical modelling have been applied extensively. In this paper, a fractional order prey–predator model is developed with the consideration of Holling type-I and Holling type-II functional response of the predator. As infection spreads through prey, the prey population is divided into two parts. In addition, we exploit the effect of harvesting to control the excessive spread of the infection. The existence and uniqueness criteria, the boundedness of the solution of the proposed model are investigated. A number of five possible equilibrium points of the proposed model are determined along with the feasibility conditions for each equilibrium points. The local stability at these equilibrium points and global stability at interior equilibrium point are investigated. Numerical simulation is presented with the help of modified Predictor-corrector method in MATLAB software to understand the dynamics of the proposed model.

Key words: Fractional order; Caputo–type derivatives; Harvesting; Local stability; Global Stability.

1. Introduction

Prey–predator models are valuable for acquiring the knowledge of the prey–predator relationship, as it acts vital role in both the theoretical and experimental ecology. Lotka [1] and Volterra [2] were the first who develops the mathematical models of prey–predator dynamics. After that, their model is extended and modified by many researchers in this research field. Kermack and McKendrick [3] who studied the transmission of infectious diseases in their model, has the great impact in the field of epidemiology. Anderson and May [4] combine the field of theoretical ecology with the epidemiology and formulated a prey–predator model with disease in population. An eco-epidemiological system with sound prey, infected prey and predator is investigated by Chattopadhyay and Arino [5]. Arino et al. [6] proposed a ratio-dependent predator–prey model with infection in prey population. A mathematical model of an infected predator-prey system with different predators' functional response is analysed by Bairagi et al. [7]. Recently, a prey-predator dynamics of disease transmission via pestamong the prey population is discussed by Das et al. [8]. Sarkar et al. [9] studied a prey-predator mathematical model with different kinds of response function. Impact of fear effect on the growth of prey in a prey-predator mathematical model is investigated by Sarkar and Khajanchi [10]. A prey–predator system with multiple delays and counter attacking strategies is analysed by Kaushik and Banerjee [11]. Panday et al. [12] studied a stage-structured prey-predator model with fear-induced group defense. There is plenty of interesting research works related to ecology and epidemiology, which are found in [13–16].

The fractional order differential equation is a superlative tool of defining the memory of numerous biological species and also and it has very close relations to the fractals. As the fractional order derivative is more realistic than ordinary differential equation, this type of models has earned popularity among the researchers. Fractional-order differential equations can be defined several ways, such as Caputo type, Riemann-Liouville type, Grunwald-Letnikov type, etc. Ahmed et al. [17] determined the Routh-Hurwitz conditions for the fractional order differential equation and their

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applications. Later, the equilibrium points, stability and numerical solutions of fractional order prey-predator and rabies model are investigated in another piece of work [18]. Panja [19] investigated the stability of a Caputo type fractional-order prey-predator model with Holling type II functional response. A Caputo fractional-order prey-predator system incorporating harvesting and the fractional response of Holling type II is studied by Mandal et al. [20]. A prey-predator system with fear effect and group defence is analysed by Das and Samanta [21] with the help of Caputo fractional differential equation. There are few works have been done on fractional differential equation [22–24].

Harvesting has a strong influence on the dynamics of predator–prey system. The dynamics of predator and prey populations with selective harvesting has been elaborated in many perspectives. A modified Leslie–Gower prey–predator model in the presence of nonlinear harvesting in prey is investigated by Gupta and Chandra [25]. A predator–prey model with square root functional response of prey with predator harvesting is analysed by Sahoo et al. [26]. Toaha and Rustam [27] studied an optimal harvesting of predator-prey model that consists of two zones, namely the free fishing and reserve zones. Dubey et al. [28] deals with a three-dimensional prey-predator model with Crowley-Martin type functional order incorporating optimal harvesting policy. A fuzzy prey–predator system consists of two preys and one predator with time delay and harvesting is investigated by Pal et al. [29]. Some authors considered harvesting for either on prey population or predator population or both predator and prey populations (for examples in [20, 30–32]).

In this paper, we construct an eco-epidemic prey-predator model incorporating harvesting effort by using some assumptions. In addition, the model is converted to a fractional order prey-predator model by using the concept of fractional order differentiation. In theoretical study section, we analyse the existence and uniqueness criteria, the boundedness of the solution of the proposed model. Five possible equilibrium points of the proposed model are determined along with the feasibility conditions for each equilibrium points. The local stability at these equilibrium points and global stability at interior equilibrium point are investigated. Numerical simulation is presented with the help of modified Predictor-corrector method [33, 34] in MATLAB software to understand the dynamics of the proposed model. The effect of harvesting effort and order of fractional order differentiation for the fractional order prey-predator model is presented elaborately.

The remaining structure of the paper is organised as follows. Some preliminaries which consists useful definition, lemmas and theorems, are presented in section 2. The detail discussion on fractional order model i.e. the mathematical formulation is described in section 3. Section 4 covers the theoretical study which contains the existence and uniqueness, boundedness, local stability and global stability of the model. Section 5 presents numerical simulation which contains graphical deliberations of the proposed model. The concluding remarks and the future scope of our work are given in section 6.

2. Some Preliminaries

In this section, we state some useful definitions, lemmas and theorems of fractional order systems to describe the analytic results of our proposed model.

Definition 1[35]: The Caputo type fractional order derivative of order $\sigma > 0$ for a function $f : C_f^n[t_0, \infty) \to \mathbb{R}$ is defined and denoted as:

$$ {}_{t_0}^{C_f}D_t^{\sigma} f(t) = \frac{1}{\Gamma(n-\sigma)} \int_{t_0}^{t} \frac{f^{(n)}(r)}{(t-r)^{\sigma-n+1}} dr $$

where, $C_f^n[t_0, \infty)$ is a space of n times continuously differentiable functions on $[t_0, \infty),\ t > t_0$ and $\Gamma(\cdot)$ is the Gamma function with $n \in \mathbb{Z}_+$ such that $\sigma \in (n-1, n)$.

In particular, for $\sigma \in (0, 1)$, above definition reduces to

$$ {}_{t_0}^{C_f}D_t^{\sigma} f(t) = \frac{1}{\Gamma(1-\sigma)} \int_{t_0}^{t} \frac{f'(r)}{(t-r)^{\sigma}} dr. $$

Lemma 1[36]: Assume that $\sigma \in (0, 1)$, $f(t) \in C_f^n[a, b]$ and ${}_{t_0}^{C_f}D_t^{\sigma} f(t)$ is continuous in $[a, b]$. If ${}_{t_0}^{C_f}D_t^{\sigma} f(t) \geq 0$ $\left({}_{t_0}^{C_f}D_t^{\sigma} f(t) \leq 0\right)$, $t \in [a, b]$ then $f(t)$ is a non-decreasing (non-increasing) function $\forall\ t \in [a, b]$.

Lemma 2[37]: Consider the system ${}_{t_0}^{C_f}D_t^{\sigma} Z(t) = \Psi(t, Z)$, $t_0 > 0$, with initial condition $Z(t_0) = Z_{t_0}$, where $\sigma \in (0, 1]$, $\Psi : [t_0, \infty) \times \Sigma \to \mathbb{R}^n$, $\Sigma \subseteq \mathbb{R}^n$, if $\Psi(t, Z)$ satisfies the local Lipschitz condition with respect to $Z \in \mathbb{R}^n$:

$$ \|\Psi(t, Z) - \Psi(t, Z_1)\| \leq K\|Z - Z_1\|, $$

then $\exists$ a unique solution on $[t_0, \infty) \times \Sigma$, where $\|Z(u_1, u_2, \ldots, u_n) - Z_1(v_1, v_2, \ldots, v_n)\| = \sum_{i=1}^{n} |u_i - v_i|,\ u_i, v_i \in \mathbb{R}$.

Lemma 3[38]: Assume that $X : [t_0, \infty) \to \mathbb{R}$ be continuous function satisfies the following:

$$ {}^{C_f}_{t_0}D^{\sigma}_t Z(t) + \mu Z(t) \le k, \quad Z(t_0) = Z_0, \quad t_0 \ge 0, $$

$\mu, k \in \mathbb{R}$

Also, for $\mu \neq 0$, $\sigma \in (0,1]$, and consider E_σ as the Mittag-Lefflar function of one parameter, we have the following inequality:

$$ Z(t) \le \left(Z_0 - \frac{k}{\mu} \right) E_\sigma[-\mu(t-t_0)^\sigma] + \frac{k}{\mu}, \ \forall\, t \ge t_0. $$

Theorem 1[39]: Let us assume that, the n-dimensional fractional differential equation system ${}^{C_f}_{t_0}D^{\sigma}_t X(t) = PX;\ X(t_0) = X_0 > 0$, where P is arbitrary constant $n \times n$ matrix and $\sigma \in (0,1)$. Then,

- ✠ If all eigenvalues $\lambda_i,\ i = 1, 2, \dots n$ of P satisfy $|\arg(\lambda_i)\| > \frac{\sigma\pi}{2}$, the solution $x = 0$ is asymptotically stable.
- ✠ If all eigenvalues of P satisfy $|\arg(\lambda_i)| \ge \frac{\sigma\pi}{2}$ and eigenvalues with $|\arg(\lambda_i)| = \frac{\sigma\pi}{2}$ have same geometric multiplicity and algebraic multiplicity, the solution $x = 0$ is stable.

Theorem 2[35]: Consider the fractional-order system of order $\sigma \in (0,1)$ such that

$$ {}^{C_f}_{t_0}D^{\sigma}_t X(t) = f(X),\ X(t_0) = X_0 > 0 $$

where, $X \in \mathbb{R}^n$. The equilibrium points of the above system are solutions to the equation $f(X) = 0$. An equilibrium point is locally asymptotically stable if all the eigenvalues λ_i of the Jacobian matrix $J = \frac{\partial f}{\partial X}$ evaluated at the equilibrium satisfy $|\arg(\lambda_i)| > \frac{\sigma\pi}{2}$.

3. Model Formulation

The total population is divided into two parts, i.e., prey population and predator population. Further, the prey population has two sub parts, i.e., susceptible prey and infected prey. In the model formulation, susceptible prey, infected prey and predator population is denoted as $S(t)$, $I(t)$ and $P(t)$ respectively at time t. To construct the mathematical model, we exploit the following assumption:

(i) The susceptible prey reproduces with logistic law and the intrinsic growth rate of susceptible prey is denoted as $\boldsymbol{M}$. Also, we consider that, $\boldsymbol{K}$ is the carrying capacity of the system.

(ii) At the infection rate $\boldsymbol{A}$, the susceptible prey becomes infected through direct contact with the infected prey, by the mass action law.

(iii) We assume the predation function of susceptible prey as $\frac{B_1 S}{(R+S)}$. This type of function is commonly known as Holling type-II functional response. Here, $\boldsymbol{B_1}$ is the maximum capturing rate of susceptible prey by the predator and $\boldsymbol{R}$ is the half saturation constant. On the other hand, the predation function of infected prey is considered as $\boldsymbol{B_2 I}$, Holling type-I functional response. Here, $\boldsymbol{B_2}$ is the maximum capturing rate of infected prey by the predator.

(iv) The conversion efficiency on susceptible prey and infected prey is measured as $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ respectively.

(v) Some of the infected prey acquire immunity at a rate $\boldsymbol{C}$ and get recovered from the infection. Therefore, some of the infected prey moves to susceptible class as they become susceptible again.

(vi) Here, the parameter F_1 and F_2 indicate the death rate of infected prey and predator respectively.

(vii) Based on CPUE (catch-per unit-effort) hypothesis [40], D_1HS and D_2HI are catch rate functions for susceptible prey and infected prey, where $\boldsymbol{H}$ is the harvesting effort and $\boldsymbol{D_i}$; $i = 1, 2$ indicates catchability coefficients of two prey species.

Combining all above assumptions the final system is as follows:

$$ \begin{cases} \dfrac{dS}{dt} = MS\left(1 - \dfrac{S}{K}\right) - ASI - \dfrac{B_1 SP}{R+S} + CI - D_1HS \\ \dfrac{dI}{dt} = ASI - B_2IP - CI - F_1I - D_2HI \\ \dfrac{dP}{dt} = \dfrac{E_1B_1SP}{R+S} + E_2B_2IP - F_2P \end{cases} \tag{1} $$

with the initial conditions, $S(0) > 0$, $I(0) > 0$, $P(0) > 0$. **Table 1** indicates the notations and units/dimensions of the parameters and variables. The schematic diagram of the model is presented in **Fig. 1**.

Table 1: Description and Units/dimensions of the variables and model parameters

Variables/ Parameters	Description	Units/ Dimensions
$S(t)$	Susceptible prey population at time t	Mass
$I(t)$	Infected prey population at time t	Mass
$P(t)$	Predator population at time t	Mass
M	Intrinsic growth rate	Mass per unit time
K	Carrying capacity	Mass
A	Rate of infection between susceptible prey and infected prey	Mass per unit time
R	Half saturation constant	Mass^{-1}
B_1	Predation rate on susceptible prey	Mass per unit time
B_2	Predation rate on infected prey	Mass per unit time
C	Recovery rate of infected prey	Mass per unit time
E_1	Conversion efficiency on susceptible prey	Dimensionless
E_2	Conversion efficiency on infected prey	Dimensionless
F_1	Death rate of infected prey	Mass per unit time
F_2	Natural death rate of predator	Mass per unit time
D_1	Catchability coefficient of susceptible prey	Mass per unit time
D_2	Catchability coefficient of infected prey	Mass per unit time
H	Harvesting effort	Mass per unit time

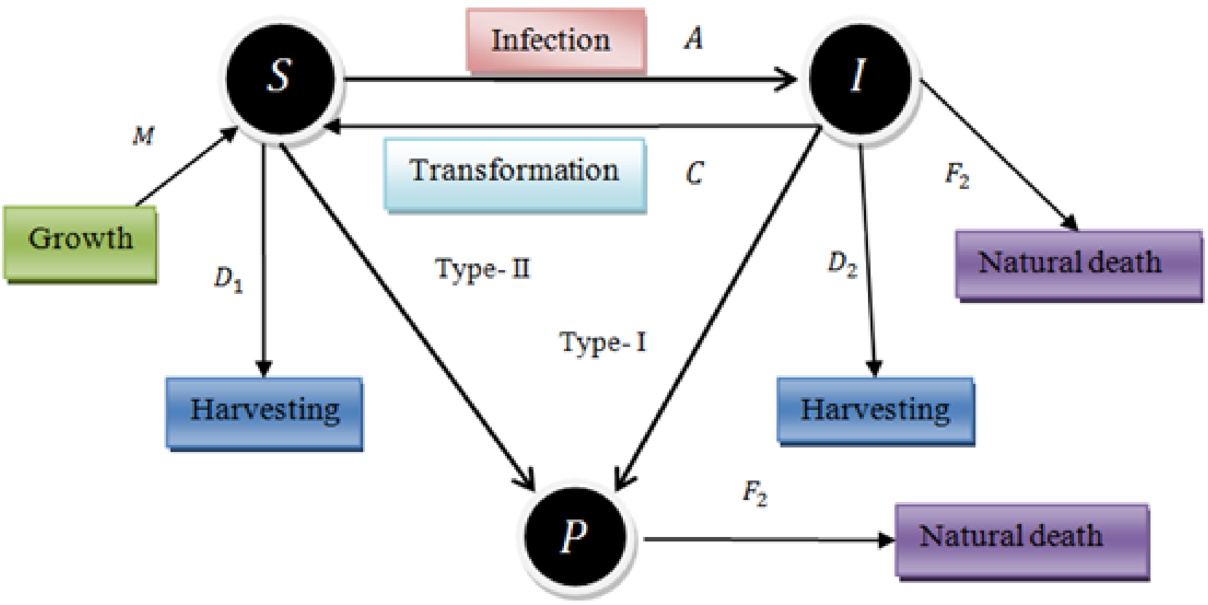

Fig. 1. Schematic diagram of the model

Now, we have proposed the fractional–order derivative of above mathematical model (1) with the help of fractional–order Caputo–type derivative [41]. Using the concept of fractional–order derivative, the model (1) becomes fractional–order prey–predator model and the model (1) reduces to the following form:

$$
\begin{cases}
{}_{t_0}^{C_f}D_t^{\sigma}S(t) = \tilde{M}S\left(1-\dfrac{S}{\tilde{K}}\right) - \tilde{A}SI - \dfrac{\tilde{B}_1SP}{\tilde{R}+S} + \tilde{C}I - \tilde{D}_1\tilde{H}S \\
{}_{t_0}^{C_f}D_t^{\sigma}I(t) = \tilde{A}SI - \tilde{B}_2IP - \tilde{C}I - \tilde{F}_1I - \tilde{D}_2\tilde{H}I \\
{}_{t_0}^{C_f}D_t^{\sigma}P(t) = \dfrac{\tilde{E}_1\tilde{B}_1SP}{\tilde{R}+S} + \tilde{E}_2\tilde{B}_2IP - \tilde{F}_2P
\end{cases} \tag{2}
$$

with initially, $S(t_0) = S_0 > 0$, $I(t_0) = I_0 > 0$ and $P(t_0) = P_0 > 0$. Also, $\sigma(0 < \sigma < 1)$ is fractional differentiation for three population classes and ${}_{t_0}^{C_f}D_t^{\sigma}$ is the fractional derivative in the sense of Caputo with initial time $t_0 \geq 0$. Since the left-hand dimension and right-hand dimension of the system of equations (2) are $time^{(-\sigma)}$ and $time^{(-1)}$ respectively, we need to modify the system (2) as

$$
\begin{cases}
{}_{t_0}^{C_f}D_t^{\sigma}S(t) = \tilde{M}^{\sigma}S\left(1-\dfrac{S}{\tilde{K}}\right) - \tilde{A}^{\sigma}SI - \dfrac{\tilde{B_1}^{\sigma}SP}{\tilde{R}+S} + \tilde{C}^{\sigma}I - \tilde{D_1}^{\sigma}\tilde{H}^{\sigma}S \\
{}_{t_0}^{C_f}D_t^{\sigma}I(t) = \tilde{A}^{\sigma}SI - \tilde{B_2}^{\sigma}IP - \tilde{C}^{\sigma}I - \tilde{F_1}^{\sigma}I - \tilde{D_2}^{\sigma}\tilde{H}^{\sigma}I \\
{}_{t_0}^{C_f}D_t^{\sigma}P(t) = \dfrac{\tilde{E_1}^{\sigma}\tilde{B_1}^{\sigma}SP}{\tilde{R}+S} + \tilde{E_2}^{\sigma}\tilde{B_2}^{\sigma}IP - \tilde{F_2}^{\sigma}P
\end{cases} \tag{3}
$$

For simplicity, we substitute parameters as $M = \tilde{M}^{\sigma}$, $K = \tilde{K}$, $A = \tilde{A}^{\sigma}$, $B_1 = \tilde{B_1}^{\sigma}$, $B_2 = \tilde{B_2}^{\sigma}$, $R = \tilde{R}$, $C = \tilde{C}^{\sigma}$, $F_1 = \tilde{F_1}^{\sigma}$, $F_2 = \tilde{F_2}^{\sigma}$, $D_1 = \tilde{D_1}^{\sigma}$, $D_2 = \tilde{D_2}^{\sigma}$, $E_1 = \tilde{E_1}^{\sigma}$, $E_2 = \tilde{E_2}^{\sigma}$, $H = \tilde{H}^{\sigma}$ and the system (3) reduces to the following form:

$$
\begin{cases}
{}_{t_0}^{C_f}D_t^{\sigma}S(t) = MS\left(1-\dfrac{S}{K}\right) - ASI - \dfrac{B_1SP}{R+S} + CI - D_1HS \\
{}_{t_0}^{C_f}D_t^{\sigma}I(t) = ASI - B_2IP - CI - F_1I - D_2HI \\
{}_{t_0}^{C_f}D_t^{\sigma}P(t) = \dfrac{E_1B_1SP}{R+S} + E_2B_2IP - F_2P
\end{cases} \tag{4}
$$

4. Theoretical Study

The existence and uniqueness, non-negativity and boundedness, equilibrium points and local stability and global stability are the most imperative and valuable part in the perspective of mathematical ecology. In the theoretical study section, the existence and uniqueness of solutions, non-negativity and boundedness, equilibrium points and local stability and global stability of the system (4) are investigated.

4.1 Existence and uniqueness of solutions

In the following theorem, the existence and uniqueness of the solutions is investigated.

Theorem 3: For every initial point $X(t_0) = (S_{t_0}, I_{t_0}, P_{t_0}) \in \mathcal{R}$, $\exists$ a unique solution $X(t) = (S(t), I(t), P(t)) \in \mathcal{R}$ of the system (4) for any time $t > t_0$.

Proof: We assume the time interval $[t_0, K_1]$, $K_1 < +\infty$ and also consider the following region:

$\mathcal{R} = \{(S, I, P) \in \mathbb{R}^3 : \max\{|S|, |I|, |P|\} \leq K_2\}$ where K_2 is finite positive real number.

Let, $T(X) = (T_1(X), T_2(X), T_3(X))$, where $X = (S, I, P)$

and $T_1(X) = MS\left(1-\dfrac{S}{K}\right) - ASI - \dfrac{B_1SP}{R+S} + CI - D_1HS$,

$T_2(X) = ASI - B_2IP - CI - F_1I - D_2HI$,

$T_3(X) = \dfrac{E_1B_1SP}{R+S} + E_2B_2IP - F_2P$

For any X, $X_1 \in \mathcal{R}$. Then

$$\|T(X) - T(X_1)\| = |T_1(X) - T_1(X_1)| + |T_2(X) - T_2(X_1)| + |T_3(X) - T_3(X_1)|$$

$$
\begin{aligned}
&= \left| M(S-S_1) - \frac{M}{K}(S^2 - S_1^2) - A(SI - S_1 I_1) - B_1 \left[\frac{SP}{R+S} - \frac{S_1 P_1}{R+S_1} \right] + C(I - I_1) - D_1 H (S - S_1) \right| \\
&\quad + |A(SI - S_1 I_1) - B_2(IP - I_1 P_1) - (C + F_1 + D_2 H)(I - I_1)| \\
&\quad + \left| E_1 B_1 \left[\frac{SP}{R+S} - \frac{S_1 P_1}{R+S_1} \right] + E_2 B_2 (IP - I_1 P_1) - F_2 (P - P_1) \right| \\
&\leq \Bigg| M(S-S_1) + \frac{2K_2 M}{K}(S - S_1) + AK_2(S-S_1) + AK_2(I - I_1) + B_1 R K_2 (S - S_1) \\
&\quad + B_1 K_2 (1 + K_2)(P - P_1) + C(I - I_1) + D_1 H (S - S_1) \Bigg| \\
&\quad + |AK_2(S - S_1) + AK_2(I - I_1) + B_2 K_2 (I - I_1) + B_2 K_2 (P - P_1) + (C + F_1 + D_2 H)(I - I_1)| \\
&\quad + |E_1 B_1 R K_2 (S - S_1) + E_1 B_1 K_2 (1 + K_2)(P - P_1) + E_2 B_2 K_2 (I - I_1) \\
&\quad + E_2 B_2 K_2 (P - P_1) + F_2 (P - P_1)| \\
&\leq \theta_1 |S - S_1| + \theta_2 |I - I_1| + \theta_3 |P - P_1| \leq \theta \|X - X_1\|
\end{aligned}
$$

where, $\theta = \max\{\theta_1, \theta_2, \theta_3\}$ and

$$
\begin{aligned}
\theta_1 &= M + \frac{2K_2 M}{K} + 2AK_2 + D_1 H + (1 + E_1) B_1 R K_2, \\
\theta_2 &= 2AK_2 + 2C + F_1 + D_2 H + (1 + E_2) B_2 K_2, \\
\theta_3 &= (1 + E_1)(1 + K_2) B_1 K_2 + (1 + E_2) B_2 K_2 + F_2.
\end{aligned}
$$

Hence, the function $T(X)$ satisfies Lipshitz's condition. Therefore, by using the Lemma 2, $\exists$ a unique solution $X(t) = (S(t),\ I(t),\ P(t)) \in \mathcal{R}$ for every initial point $X(t_0) = (S_{t_0}, I_{t_0}, P_{t_0}) \in \mathcal{R}$, $\forall\ t > t_0$.

4.2 Non-negativity and boundedness

In the following theorem, the non-negativity and boundedness of the system (4) is investigated.

Theorem 4: The solutions of system (4) which originate in $\mathbb{R}^3_+$ are non-negative and bounded uniformly if $E_2 < E_1$.

Proof: To prove non-negativity of the system (4), we consider that

$$\mathcal{R}_+ = \{(S, I, P) \in \mathcal{R} : S, I, P \in \mathbb{R}_+\},$$

where $\mathbb{R}_+$ indicates the set of all real numbers greater than or equals to zero. Let, $X(t_0) = (S_{t_0}, I_{t_0}, P_{t_0}) \in \mathcal{R}_+$ be the initial solution of the system (4). Suppose ζ is a real number satisfying $t_0 \leq t < \zeta$ such that

$$
\begin{cases}
S(t) > 0, & t_0 \leq t < \zeta \\
S(t) = 0, & t = \zeta \\
S(t) < 0, & t = \zeta^+
\end{cases}
$$

From the first equation of the system (4), we have ${}^{C_f}_{t_0}D^\sigma_t S(t)\big|_{S(\zeta)=0} = 0$. By Lemma 1, we have $S(\zeta^+) = 0$, which contradicts $S(\zeta^+) < 0$. Hence, we have $S(t) \geq 0$, $\forall\ t \in [t_0, \infty]$. By similar manner, we can prove that $I(t) \geq 0$ and $P(t) \geq 0 \forall\ t \in [t_0, \infty]$.

Next, to prove the boundedness, let us consider the function

$$W(t) = S(t) + I(t) + \frac{1}{E_1} P(t).$$

Then, for $E_2 < E_1$ we have

$$
\begin{aligned}
{}^{C_f}_{t_0}D^\sigma_t W(t) &= MS\left(1 - \frac{S}{K}\right) + \left(\frac{E_2}{E_1} - 1\right) B_2 IP \\
&\quad - D_1 HS - D_2 HI - F_1 I - \frac{F_2}{E_1} P \\
&\leq MS\left(1 - \frac{S}{K}\right)
\end{aligned}
$$

Then for arbitrarily chosen η, we have

$$
\begin{aligned}
{}^{C_f}_{t_0}D^\sigma_t W(t) + \eta W &\leq MS\left(1 - \frac{S}{K} + \frac{\eta}{M}\right) \\
&\qquad\qquad \text{since } S(t) > 0 \\
&\leq -\frac{M}{K}\left(S - \frac{K(M+\eta)}{2M}\right)^2 + \frac{K(M+\eta)^2}{4M} \\
&\leq \frac{K(M+\eta)^2}{4M}.
\end{aligned}
$$

with the help of the Lemma 3, we have

$$W(t) \leq \left(W(t_0) - \frac{K(M+\eta)^2}{4M\eta}\right) E_\sigma[-\eta(t-t_0)^\sigma] + \frac{K(M+\eta)^2}{4M\eta} \to \frac{K(M+\eta)^2}{4M\eta} \text{ as } t \to \infty.$$

Therefore, for $E_2 < E_1$, the solutions of system (4) starting in $\mathcal{R}_+$ are lying in the region

$$\Delta = \left\{(S, I, P) \in \mathcal{R}_+ : S + I + \frac{1}{E_1} P(t) \leq \frac{K(M+\eta)^2}{4M\eta} + \epsilon,\ \epsilon > 0\right\}.$$

4.3 Equilibrium points and local stability analysis

We investigate the existence of five equilibrium points namely the trivial equilibrium, the boundary or axial equilibrium, the infection free equilibrium, the predator free equilibrium and the interior equilibrium. We use some theorems to investigate local stability of the system (4) for five equilibrium points.

Theorem 5: The trivial equilibrium point $E_0(0,0,0)$ is a saddle point if $M > D_1H$ and stable or else.

Proof: The Jacobian matrix at the trivial equilibrium $E_0 = (0,0,0)$ is given by

$$J(E_0) = \begin{pmatrix} M - D_1H & C & 0 \\ 0 & -(C + F_1 + D_2H) & 0 \\ 0 & 0 & -F_2 \end{pmatrix}$$

Therefore, all eigenvalues of $J(E_0)$ are $\lambda_1 = M - D_1H$, $\lambda_2 = -(C + F_1 + D_2H)$, $\lambda_3 = -F_2$.
Here, λ_1 will be negative if $M < D_1H$, when $|\arg(\lambda_1)| = \pi > \frac{\sigma\pi}{2}$ and λ_1 will be positive if $M > D_1H$, when $|\arg(\lambda_1)| = 0 < \frac{\sigma\pi}{2}$. Clearly, other eigenvalues are negative. Therefore,

$$|\arg(\lambda_i)| = \pi > \frac{\sigma\pi}{2}, \text{ for } i = 2, 3.$$

Hence, the trivial equilibrium point is a saddle point if $M > D_1H$. Therefore, E_0 is unstable. If $M < D_1H$, E_0 become stable and all the population are going to extinct.

Theorem 6: The boundary or axial equilibrium point $E_1(K,0,0)$ is locally asymptotically stable if $AK < C + F_1 + D_2H$, $\frac{E_1B_1K}{R+K} < F_2$ and unstable otherwise.

Proof: The Jacobian matrix at the boundary equilibrium $E_1 = (K,0,0)$ is given by

$$J(E_1) = \begin{pmatrix} -M - D_1H & -AK + C & -\frac{B_1K}{R+K} \\ 0 & AK - C - F_1 - D_2H & 0 \\ 0 & 0 & \frac{E_1B_1K}{R+K} - F_2 \end{pmatrix}$$

The characteristic equation at $E_1(K,0,0)$ is

$$(\lambda + M + D_1H)(\lambda - AK + C + F_1 + D_2H)\left(\lambda - \frac{E_1B_1K}{R+K} + F_2\right) = 0.$$

The eigenvalues of $J(E)_1$ are $\lambda_1 = -M - D_1H$, $\lambda_2 = AK - C - F_1 - D_2H$, $\lambda_3 = \frac{E_1B_1K}{R+K} - F_2$.
Clearly, $|\arg(\lambda_i)| = \pi > \frac{\theta\pi}{2}$, for $i = 1,2,3$ when $AK < C + F_1 + D_2H$ and $\frac{E_1B_1K}{R+K} < F_2$ holds. Therefore, the boundary equilibrium E_1 is locally asymptotically stable if $AK < C + F_1 + D_2H$, $\frac{E_1B_1K}{R+K} < F_2$ and unstable otherwise.

Theorem 7: The infection free equilibrium $E_2(S_1, 0, P_1)$ is locally asymptotically stable if
(a) $AS_1 < B_2P_1 + C + F_1 + D_2H$ and
(b) $M + \frac{E_1B_1S_1}{R+S_1} < \frac{2MS_1}{K} + \frac{RB_1P_1}{(R+S_1)^2} + D_1H + F_2$
where, $S_1 = \frac{F_2R}{E_1B_1 - F_2}$ and $P_1 = \frac{R+S_1}{B_1}\left[M\left(1 - \frac{S_1}{K}\right) - D_1H\right]$, with $E_1B_1 > F_2$ and $M\left(1 - \frac{S_1}{K}\right) > D_1H$.

Proof: By solving the first and last equation of the system (4) at the point $E_2 = (S_1, 0, P_1)$, we have

$$S_1 = \frac{F_2R}{E_1B_1 - F_2}$$

$$\text{and } \ P_1 = \frac{R+S_1}{B_1}\left[M\left(1 - \frac{S_1}{K}\right) - D_1H\right].$$

The equilibrium E_2 is feasible if $E_1B_1 > F_2$ and $M\left(1 - \frac{S_1}{K}\right) > D_1H$.

Now, the Jacobian matrix for $E_2(S_1,0,P_1)$ is given by,

$$J(E_2)=\begin{pmatrix} M-\frac{2MS_1}{K}-\frac{RB_1P_1}{(R+S_1)^2}-D_1H & -AS_1+C & -\frac{B_1S_1}{R+S_1}\\ 0 & AS_1-C-B_2P_1-F_1-D_2H & 0\\ \frac{RE_1B_1P_1}{(R+S_1)^2} & E_2B_2P_1 & \frac{E_1B_1S_1}{R+S_1}-F_2\end{pmatrix}$$

The characteristic equation at $E_2(S_1,0,P_1)$ is

$$(AS_1-C-B_2P_1-F_1-D_2H-\lambda)\left(\lambda^2-\lambda\left(M-\frac{2MS_1}{K}-\frac{RB_1P_1}{(R+S_1)^2}-D_1H+\frac{E_1B_1S_1}{R+S_1}-F_2\right)\right.$$
$$\left.+\left(M-\frac{2MS_1}{K}-\frac{RB_1P_1}{(R+S_1)^2}-D_1H\right)\left(\frac{E_1B_1S_1}{R+S_1}-F_2\right)+\frac{B_1S_1}{R+S_1}\right)=0.$$

The eigenvalues of the above Jacobian matrix for $E_2(S_1,0,P_1)$ is $\lambda_1=AS_1-C-B_2P_1-F_1-D_2H$ and other two eigenvalues can be evaluated from the equation

$$\lambda^2-\lambda\left(M-\frac{2MS_1}{K}-\frac{RB_1P_1}{(R+S_1)^2}-D_1H+\frac{E_1B_1S_1}{R+S_1}-F_2\right)$$
$$+\left(M-\frac{2MS_1}{K}-\frac{RB_1P_1}{(R+S_1)^2}-D_1H\right)\left(\frac{E_1B_1S_1}{R+S_1}-F_2\right)+\frac{B_1S_1}{R+S_1}=0.$$

Clearly, $|\arg(\lambda_i)|=\pi>\frac{\theta\pi}{2}$, for $i=1,2,3$ when
(a) $AS_1<B_2P_1+C+F_1+D_2H$ and
(b) $M+\frac{E_1B_1S_1}{R+S_1}<\frac{2MS_1}{K}+\frac{RB_1P_1}{(R+S_1)^2}+D_1H+F_2$.
Therefore, the equilibrium E_2 is locally asymptotically stable if above two condition holds and unstable otherwise.

Theorem 8: The predator free equilibrium $E_3(S_2,I_2,0)$ is locally asymptotically stable if
(a) $M<\frac{2MS_2}{K}+AI_2+D_1H$ and
(b) $\frac{E_1B_1S_2}{R+S_2}+E_2B_2I_2<F_2$
where, $S_2=\frac{C+F_1+D_2H}{A}$
and $I_2=\frac{(C+F_1+D_2H)[M(KA-C-F_1-D_2H)-D_1HKA]}{A^2(F_1+D_2H)}$,
with $M(KA-C-F_1-D_2H)>D_1HKA$.

Proof: By solving the first and second equation of the system (4) at the point $E_3=(S_2,I_2,0)$, we have

$$S_2=\frac{C+F_1+D_2H}{A}\quad\text{and}\quad I_2=\frac{(C+F_1+D_2H)[M(KA-C-F_1-D_2H)-D_1HKA]}{A^2(F_1+D_2H)}.$$

The equilibrium E_3 is feasible if $M(KA-C-F_1-D_2H)>D_1HKA$.
Now, the Jacobian matrix for $E_3(S_2,I_2,0)$ is given by,

$$J(E_3)=\begin{pmatrix} M-\frac{2MS_2}{K}-AI_2-D_1H & -AS_2+C & -\frac{B_1S_2}{R+S_2}\\ AI_2 & 0 & -B_2I_2\\ 0 & 0 & \frac{E_1B_1S_2}{R+S_2}+E_2B_2I_2-F_2\end{pmatrix}$$

The characteristic equation at $E_3(S_2,I_2,0)$ is

$$\left(\frac{E_1B_1S_2}{R+S_2}+E_2B_2I_2-F_2-\lambda\right)\left[\lambda^2-\lambda\left(M-\frac{2MS_2}{K}-AI_2-D_1H\right)+AI_2(AS_2-C)\right]=0.$$

The eigenvalues of the above Jacobian matrix for $E_3(S_2,I_2,0)$ is

$$\lambda_1=\frac{E_1B_1S_2}{R+S_2}+E_2B_2I_2-F_2$$

and other two eigenvalues can be evaluated from the equation

$$\lambda^2-\lambda\left(M-\frac{2MS_2}{K}-AI_2-D_1H\right)$$
$$+AI_2(AS_2-C)=0.$$

Clearly, $|\arg(\lambda_i)| = \pi > \frac{\theta\pi}{2}$, for $i = 1, 2, 3$ when

(a) $M < \frac{2MS_2}{K} + AI_2 + D_1 H$ and

(b) $\frac{E_1 B_1 S_2}{R+S_2} + E_2 B_2 I_2 < F_2$.

Therefore, the equilibrium E_3 is locally asymptotically stable if above two condition holds and unstable otherwise.

Theorem 9: The interior equilibrium $E^*(S^*, I^*, P^*)$ is conditionally locally asymptotically stable, where $S^* = \frac{C+F_1+D_2H+B_2P^*}{A}$, $I^* = \frac{1}{E_2B_2}\left[F_2 - \frac{E_1B_1S^*}{R+S^*}\right]$ with $F_2 > \frac{E_1B_1S^*}{R+S^*}$ and P^* can be determined from the first equation of the system (4) at E^*.

Proof: From the equations given in system (4),

$$S^* = \frac{C + F_1 + D_2H + B_2P^*}{A}$$

and $$I^* = \frac{1}{E_2B_2}\left[F_2 - \frac{E_1B_1S^*}{R+S^*}\right].$$

If we put the value of S^* and I^* in the above system of equations, then we get the value of P^*. The equilibrium E^* is feasible if $F_2 > \frac{E_1B_1S^*}{R+S^*}$.

The Jacobian matrix for $E^*(S^*, I^*, P^*)$ is given by,

$$J(E^*) = \begin{pmatrix} M - \frac{2MS^*}{K} - AI^* - \frac{RB_1P^*}{(R+S^*)^2} - D_1H & -AS^* + C & -\frac{B_1S^*}{a+S^*} \\ AI^* & AS^* - C - B_2P^* - F_1 - D_2H & -B_2I^* \\ \frac{RE_1B_1P^*}{(R+S^*)^2} & E_2B_2P^* & \frac{E_1B_1S^*}{R+S^*} + E_2B_2I^* - F_2 \end{pmatrix}$$

The characteristic equation of this system around E^* can be written as,

$$\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 = 0 \qquad (5)$$

The values of Ω_1, Ω_2 and Ω_3 can be determined from the Jacobian matrix $J(E^*)$, where

$$\begin{aligned} \Omega_1 &= -(A_{11} + A_{22} + A_{33}), \\ \Omega_2 &= [(A_{22}A_{33} + A_{11}A_{33} + A_{11}A_{22}) \\ &\qquad - (A_{23}A_{32} + A_{13}A_{31} + A_{12}A_{21})], \\ \Omega_3 &= -[(A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} \\ &\qquad + A_{13}A_{21}A_{32}) - (A_{11}A_{23}A_{32} \\ &\qquad + A_{12}A_{21}A_{33} + A_{13}A_{22}A_{31})] \end{aligned}$$

where, $A_{11} = M - \frac{2MS^*}{K} - AI^* - \frac{RB_1P^*}{(R+S^*)^2} - D_1H$, $A_{12} = -AS^* + C$, $A_{13} = -\frac{B_1S^*}{a+S^*}$, $A_{21} = AI^*$, $A_{22} = AS^* - C - B_2P^* - F_1 - D_2H$, $A_{23} = -B_2I^*$, $A_{31} = \frac{RE_1B_1P^*}{(R+S^*)^2}$, $A_{32} = E_2B_2P^*$, $A_{33} = \frac{E_1B_1S^*}{R+S^*} + E_2B_2I^* - F_2$.

Let, Δ be the discriminant of polynomial (5) and it can be expressed as follows

$$\Delta = 18\Omega_1\Omega_2\Omega_3 + \Omega_1^2\Omega_2^2 - 4\Omega_1^2\Omega_3 - 4\Omega_2^2 - 27\Omega_3^2.$$

Clearly, interior equilibrium point E^* is locally asymptotically stable if any one of the following condition holds [17]:

- ✠ $\Delta > 0$, $\Omega_1 > 0$, $\Omega_3 > 0$ and $\Omega_1\Omega_2 > \Omega_3$,
- ✠ $\Delta < 0$, $\Omega_1 \geq 0$, $\Omega_2 \geq 0$, $\Omega_3 > 0$ and $\sigma > \dfrac{2}{3}$,
- ✠ $\Delta < 0$, $\Omega_1 > 0$, $\Omega_2 > 0$, $\Omega_1\Omega_2 = \Omega_3$ and $\sigma \in (0, 1]$.

4.4 Global stability analysis

In this section, the global stability of the interior equilibrium $E^*(S^*, I^*, P^*)$ is investigated through the following theorem.

Theorem 10: Assume that $E^*(S^*, I^*, P^*)$ is locally asymptotically stable. If $M(0) > 0$, then it is globally asymptotically stable, where $-\Theta(S) = -\frac{RME_1}{K(R+S^*)} + \frac{RE_1B_1P^*}{(R+S)(R+S^*)^2} - \frac{RE_1CI^*}{S^{*2}(R+S^*)}$ in the region $\Lambda = \left\{(S, I, P) : \frac{S}{S^*} > 1\right\}$.

Proof: We construct a Lyapunov function in the following way:

$$\begin{aligned} \text{i.e., } V_f = k_1 &\left\{(S - S^*) - S^* \ln\frac{S}{S^*}\right\} \\ &+ k_2\left\{(I - I^*) - I^* \ln\frac{I}{I^*}\right\} \\ &+ k_3\left\{(P - P^*) - P^* \ln\frac{P}{P^*}\right\}, \end{aligned}$$

where, k_1, k_2 and k_3 are positive constants to be chosen suitably.

Taking time derivative of V_f, we get

$$\begin{aligned}
{}^{C_f}_{t_0}D^{\sigma}_{t}V_f(t) &= k_1\frac{S-S^*}{S}\,{}^{C_f}_{t_0}D^{\sigma}_{t}S(t) + k_2\frac{I-I^*}{I}\,{}^{C_f}_{t_0}D^{\sigma}_{t}I(t) + k_3\frac{P-P^*}{P}\,{}^{C_f}_{t_0}D^{\sigma}_{t}P(t)\\
&= k_1(S-S^*)\left[M\left(1-\frac{S}{K}\right) - AI + C\frac{I}{S} - \frac{B_1P}{R+S} - D_1H\right]\\
&\quad + k_2(I-I^*)[AS - B_2P - C - F_1 - D_2H] + k_3(P-P^*)\left[\frac{E_1B_1S}{R+S} + E_2B_2I - F_2\right]\\
&= k_1(S-S^*)\left[-\frac{M}{K}(S-S^*) - A(I-I^*) + C\left(\frac{I}{S}-\frac{I^*}{S^*}\right) - B_1\left(\frac{P}{R+S}-\frac{P^*}{R+S^*}\right)\right]\\
&\quad + k_2(I-I^*)[A(S-S^*) - B_2(P-P^*)]\\
&\quad + k_3(P-P^*)\left[E_1B_1\left(\frac{S}{R+S}-\frac{S^*}{R+S^*}\right) + E_2B_2(I-I^*)\right]\\
&= -\left(\frac{Mk_1}{K} - \frac{B_1k_1P^*}{(R+S)(R+S^*)} + \frac{Ck_1I^*}{SS^*}\right)(S-S^*)^2\\
&\quad + \left(-\frac{Mk_1}{K} - Ak_1 + Ak_2 + \frac{Ck_1}{S}\right)(S-S^*)(I-I^*)\\
&\quad + (-B_2k_2 + E_2B_2k_3)(I-I^*)(P-P^*)\\
&\quad + \left(-\frac{B_1k_1}{R+S} + \frac{RE_1B_1k_3}{(R+S)(R+S^*)}\right)(S-S^*)(P-P^*)\\
&\le -\left(\frac{Mk_1}{K} - \frac{B_1k_1P^*}{(R+S)(R+S^*)} + \frac{Ck_1I^*}{S^{*2}}\right)(S-S^*)^2\\
&\quad + \left(-\frac{Mk_1}{K} - Ak_1 + Ak_2 + \frac{Ck_1}{S^*}\right)(S-S^*)(I-I^*)\\
&\quad + (-B_2k_2 + E_2B_2k_3)(I-I^*)(P-P^*)\\
&\quad + \left(-\frac{B_1k_1}{R+S} + \frac{RE_1B_1k_3}{(R+S)(R+S^*)}\right)(S-S^*)(P-P^*) \quad \text{where } \frac{S}{S^*} > 1
\end{aligned}$$

Now, taking $k_1 = \dfrac{RE_1}{R+S^*}$, $k_2 = E_2$, $k_3 = 1$ and with the relation $k_1\left(\dfrac{M}{K} + A - \dfrac{C}{S^*}\right) = Ak_2$.

Therefore, we have

$$\begin{aligned}
{}^{C_f}_{t_0}D^{\sigma}_{t}V_f(t) &\le -\left(\frac{Mk_1}{K} - \frac{B_1k_1P^*}{(R+S)(R+S^*)} + \frac{Ck_1I^*}{S^{*2}}\right)(S-S^*)^2\\
&= -\left(\frac{RME_1}{K(R+S^*)} - \frac{RE_1B_1P^*}{(R+S)(R+S^*)^2} + \frac{RE_1CI^*}{S^{*2}(R+S^*)}\right)(S-S^*)^2\\
&= -\Theta(S)(S-S^*)^2
\end{aligned}$$

$$\begin{aligned}
\text{where, } -\Theta(S) &= -\frac{RME_1}{K(R+S^*)} + \frac{RE_1B_1P^*}{(R+S)(R+S^*)^2} - \frac{RE_1CI^*}{S^{*2}(R+S^*)}\\
&\le -\frac{arc_1}{K(a+S^*)} + \frac{c_1p_1P^*}{(a+S^*)^2} - \frac{ac_1\beta I^*}{S^{*2}(a+S^*)}\\
&= -\Theta(0)
\end{aligned}$$

Thus, if $\Theta(0) > 0$, then ${}^{C_f}_{t_0}D^{\sigma}_{t}V_f(t) \le 0$.

Therefore, $E^*(S^*, I^*, P^*)$ is globally asymptotically stable.

5. Numerical Simulations

Numerical simulations have been executed with the help of modified Predictor–corrector method [33, 34] in MATLAB R2014a software package. To perform the numerical simulations of the system (4), the parametric values are considered as $M = 2.0$, $K = 50$, $R = 0.2$, $B_1 = 0.3$, $B_2 = 0.6$, $F_1 = 0.18$, $F_2 = 0.22$, $E_1 = 0.4$, $E_2 = 0.15$, $C = 0.01$, $D_1 = 2$, $D_2 = 2$, $A \in (0.1, 0.9)$, $H \in (0, 0.8)$ and order of the fractional differentiation $\sigma \in (0, 1]$.

To draw the **Fig. 2**–**Fig. 5**, we use the values of parameter $M = 2.0$, $K = 50$, $R = 0.2$, $B_1 = 0.3$, $B_2 = 0.6$, $F_1 = 0.18$, $F_2 = 0.22$, $E_1 = 0.4$, $E_2 = 0.15$, $C = 0.01$, $D_1 = 2$, $D_2 = 2$ and $A = 0.8$ in the absence of harvesting effort, i.e., $H = 0.0$. For **Fig. 2** and **Fig. 3**, we use order of the fractional differentiation $\sigma = 0.99$ and $\sigma = 0.85$ with the initial condition $(S_0, I_0, P_0) = (2, 0.5, 1)$. In both cases we see that stable solution curves but it converges to different equilibrium point, which are $E^* = (17.56, 1.92, 23.48)$ and $E^* = (16.85, 1.93, 22.72)$ respectively. **Fig. 4** represents the variation in population curves for three different values of σ. Blue lines, green lines and red lines indicate the population curves for $\sigma = 0.99$, $\sigma = 0.85$ and $\sigma = 0.45$ respectively. **Fig. 5** represents the three dimensional phase portrait for $\sigma = 0.99$ and $H = 0.0$ with three different initial conditions. Blue line, green line and red line indicates the population curves for initial conditions $I_1 = (12, 5, 8)$, $I_2 = (6, 2, 6)$ and $I_3 = (2, 0.5, 1)$ respectively. For the Fig. 5, it can be concluded that the system (4) is globally asymptotically stable for $\sigma = 0.99$ and $H = 0.0$.

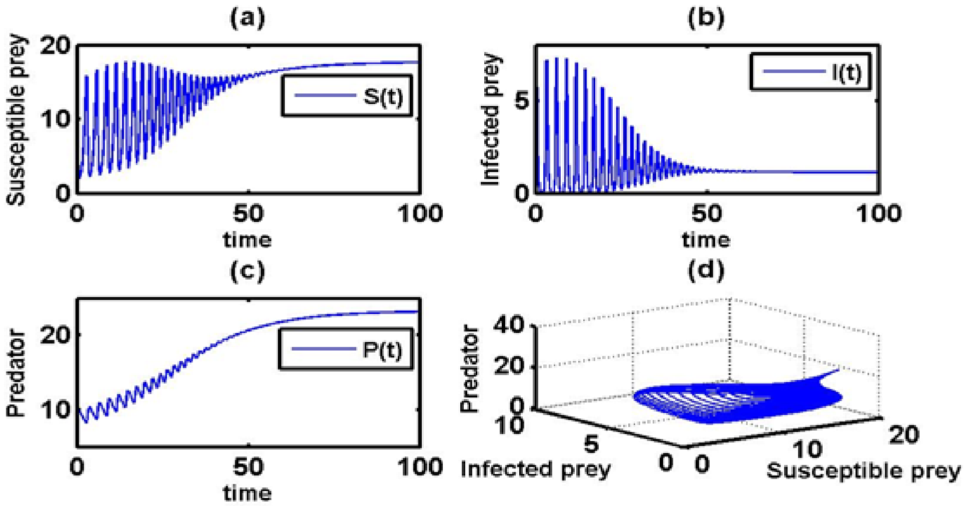

Fig. 2. Time series and phase portrait of the system (4) around $E^* = (17.56, 1.92, 23.48)$ for $\sigma = 0.99$ and $H = 0.0$.

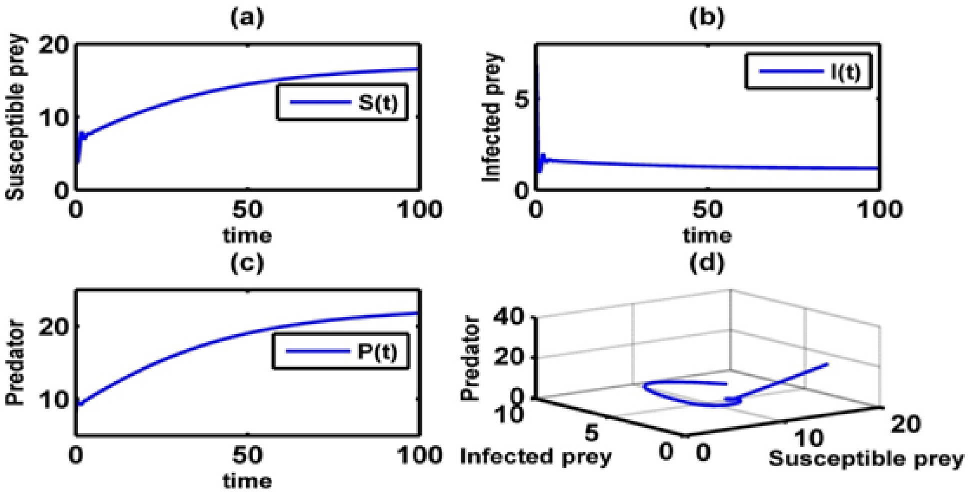

Fig. 3. Time series and phase portrait of the system (4) around $E^* = (16.85, 1.93, 22.72)$ for $\sigma = 0.85$ and $H = 0.0$.

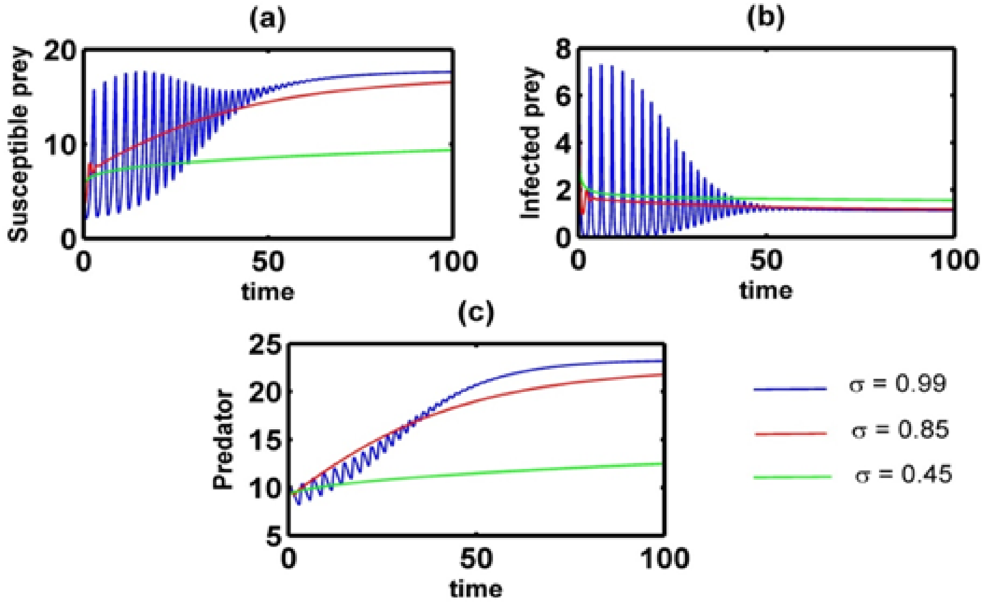

Fig. 4. Time series plot of the system (4) at $H = 0.0$ for different values of σ: (a) $\sigma = 0.99$ (blue line), (b) $\sigma = 0.85$ (red line) and (c) $\sigma = 0.45$ (green line).

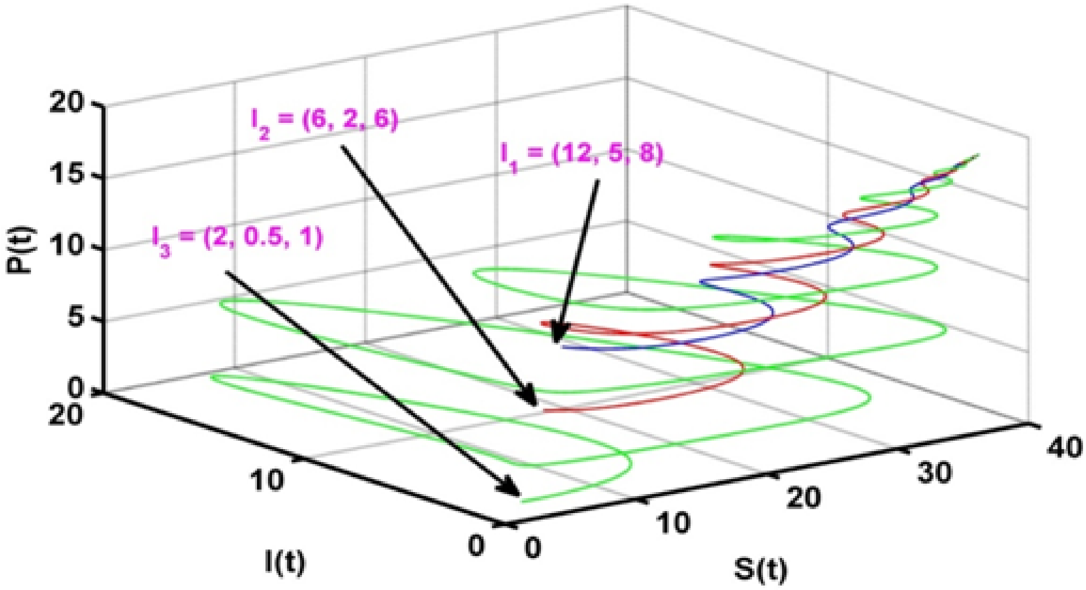

Fig. 5. Three dimensional phase portrait of the system (4) at $H = 0.0$ for different initial conditions $I_1 = (12, 5, 8)$ (blue line), $I_2 = (6, 2, 6)$ (red line) and $I_3 = (2, 0.5, 1)$ (green line).

To draw the **Fig. 6**–**Fig. 9**, we use the values of parameter $M = 2.0$, $K = 50$, $R = 0.2$, $B_1 = 0.3$, $B_2 = 0.6$, $F_1 = 0.18$, $F_2 = 0.22$, $E_1 = 0.4$, $E_2 = 0.15$, $C = 0.01$, $D_1 = 2$, $D_2 = 2$ and $A = 0.3$ in the presence of harvesting effort, i.e., $H = 0.4$. For **Fig. 6** and **Fig. 7**, we use order of the fractional differentiation $\sigma = 0.99$ and $\sigma = 0.85$ with the initial condition $(S_0, I_0, P_0) = (2, 0.5, 1)$. In both cases we see that stable solution curves and it converges to the almost same equilibrium point, which is $E^* = (17.22, 1.55, 7.68)$. **Fig. 8** represents the variation in population curves for three different values of σ. Blue lines, green lines and red lines indicate the population curves for $\sigma = 0.99$, $\sigma = 0.85$ and $\sigma = 0.45$ respectively. From this figure, the variation in population curve and equilibrium point is clearly visible for $\sigma = 0.45$. **Fig. 9** represents the three dimensional phase portrait for $\sigma = 0.99$ and $H = 0.4$ with three different initial conditions. Blue line, green line and red line indicates the population curves for initial conditions $I_1 = (12, 5, 8)$, $I_2 = (6, 2, 6)$ and $I_3 = (2, 0.5, 1)$ respectively. For the Fig. 9, it can be concluded that the system (4) is globally asymptotically stable for $\sigma = 0.99$ and $H = 0.4$.

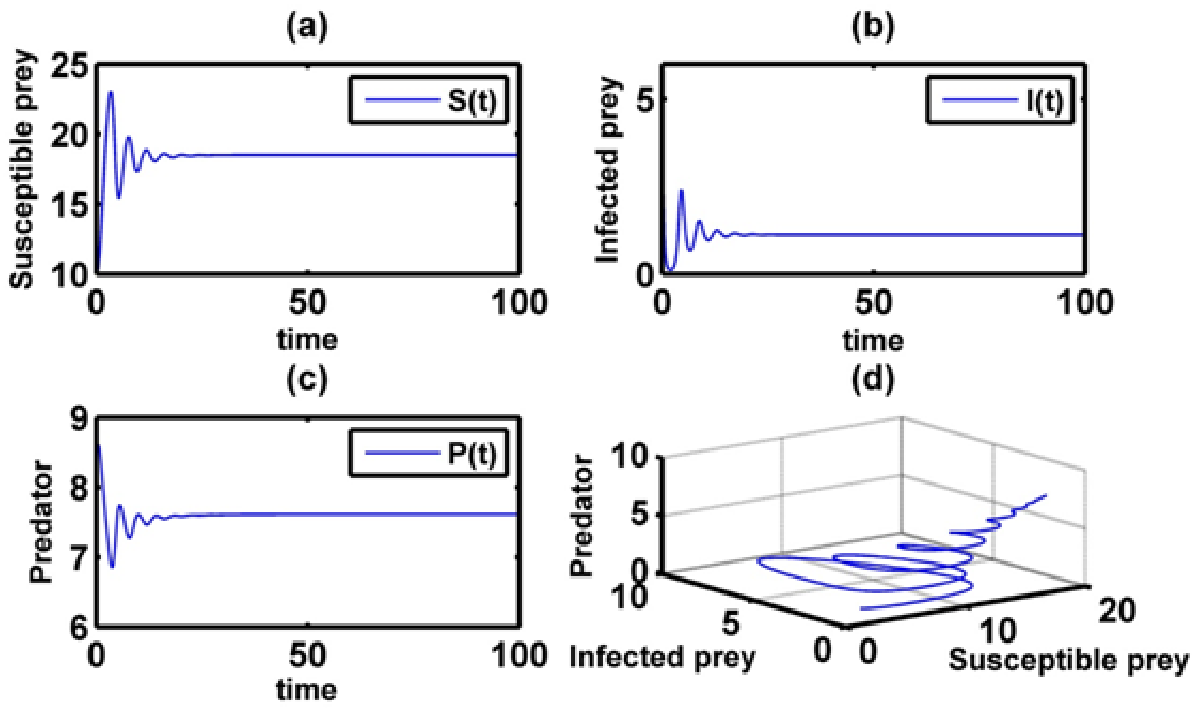

Fig. 6. Time series and phase portrait of the system (4) around $E^* = (17.56, 1.92, 23.48)$ for $\sigma = 0.99$ and $H = 0.4$.

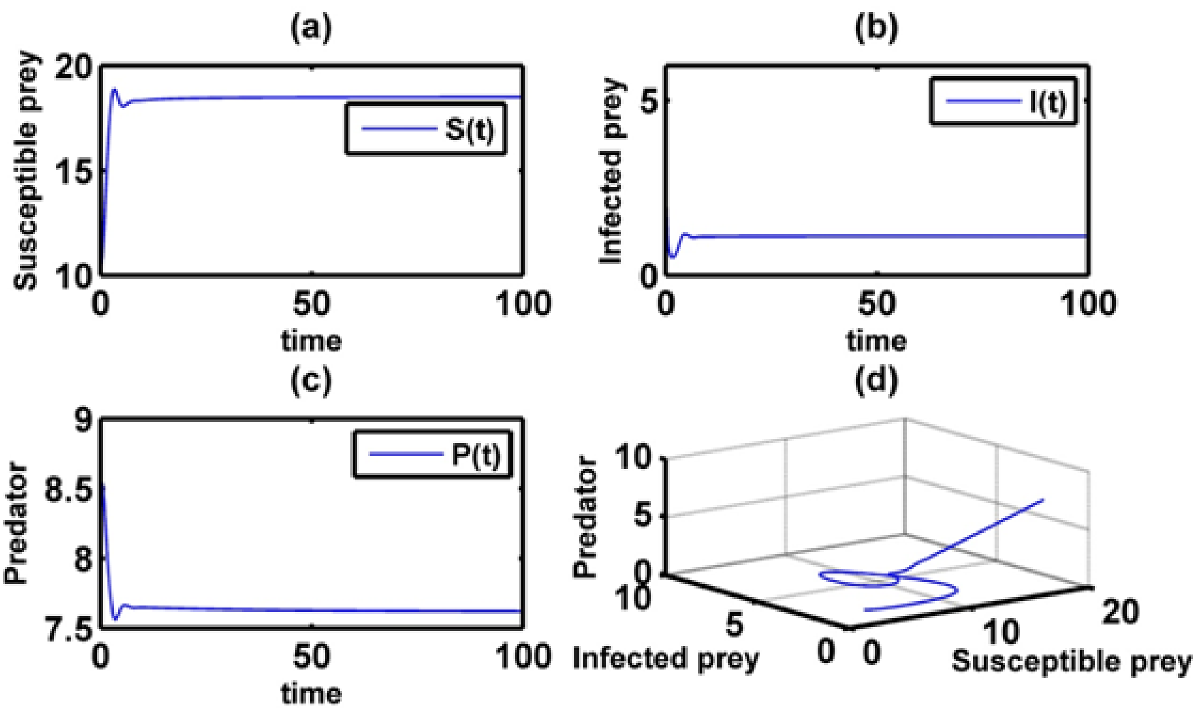

Fig. 7. Time series and phase portrait of the system (4) around $E^* = (16.85, 1.93, 22.72)$ for $\sigma = 0.85$ and $H = 0.4$.

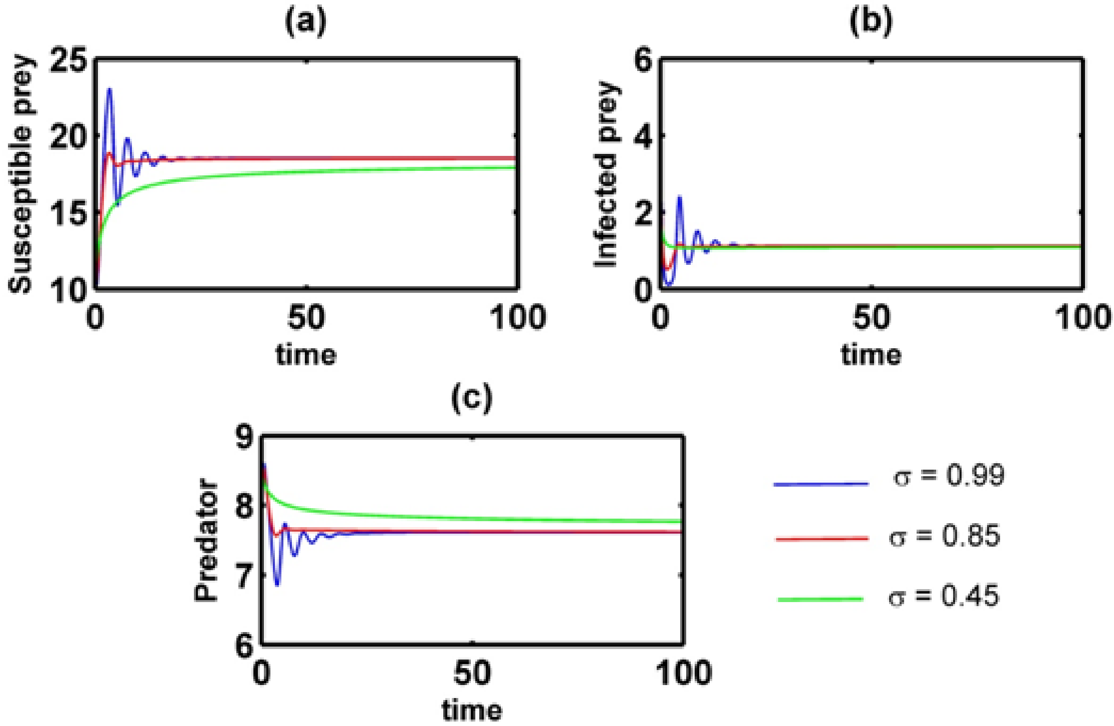

Fig. 8. Time series plot of the system (4) at $H = 0.4$ for different values of σ: (a) $\sigma = 0.99$ (blue line), (b) $\sigma = 0.85$ (red line) and (c) $\sigma = 0.45$ (green line).

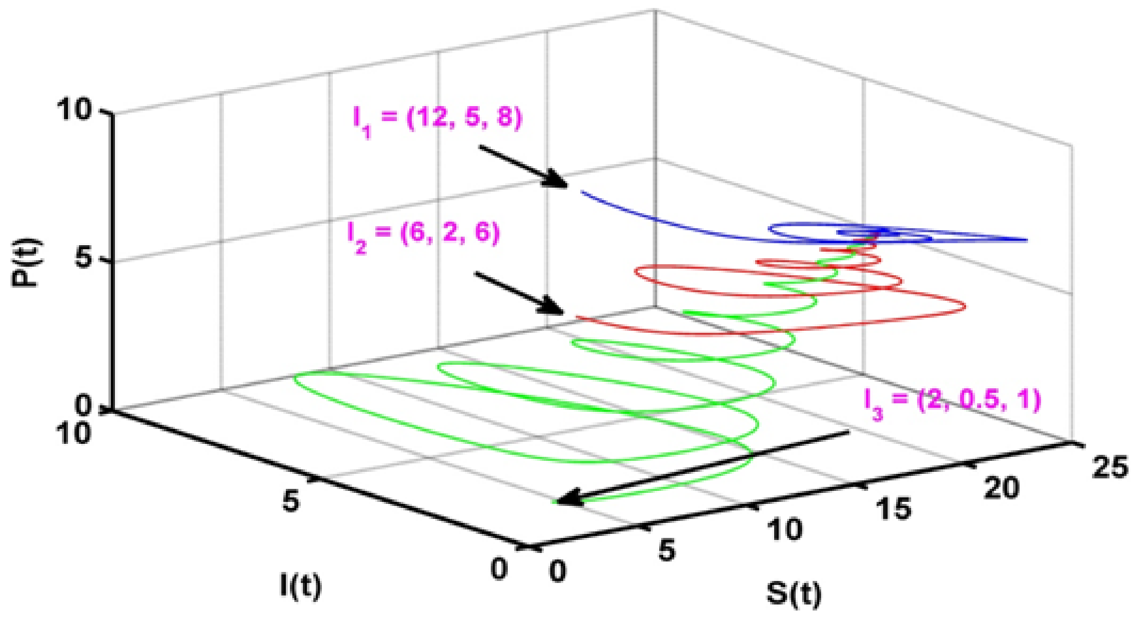

Fig. 9. Three dimensional phase portrait of the system (4) at $H = 0.4$ for different initial conditions $I_1 = (12, 5, 8)$ (blue line), $I_2 = (6, 2, 6)$ (red line) and $I_3 = (2, 0.5, 1)$ (green line).

To draw the **Fig. 10**–**Fig. 12**, we use the values of parameter $M = 2.0$, $K = 50$, $R = 0.2$, $B_1 = 0.3$, $B_2 = 0.6$, $F_1 = 0.18$, $F_2 = 0.22$, $E_1 = 0.4$, $E_2 = 0.15$, $C = 0.01$, $D_1 = 2$, $D_2 = 2$ and $A = 0.3$ with the initial condition $(S_0, I_0, P_0) = (2, 0.5, 1)$. Fig. 10, Fig. 11 and Fig. 12 represent the time series plot and phase portrait for the different values of H at $\sigma = 0.99$, $\sigma = 0.85$ and $\sigma = 0.45$ respectively. Blue lines, green lines and red lines indicate the population curves for $H = 0.6$, $H = 0.4$ and $H = 0.2$ respectively. From these figures it is clearly visible that as harvesting effort increases, population of the susceptible prey, infected prey and predator decreases.

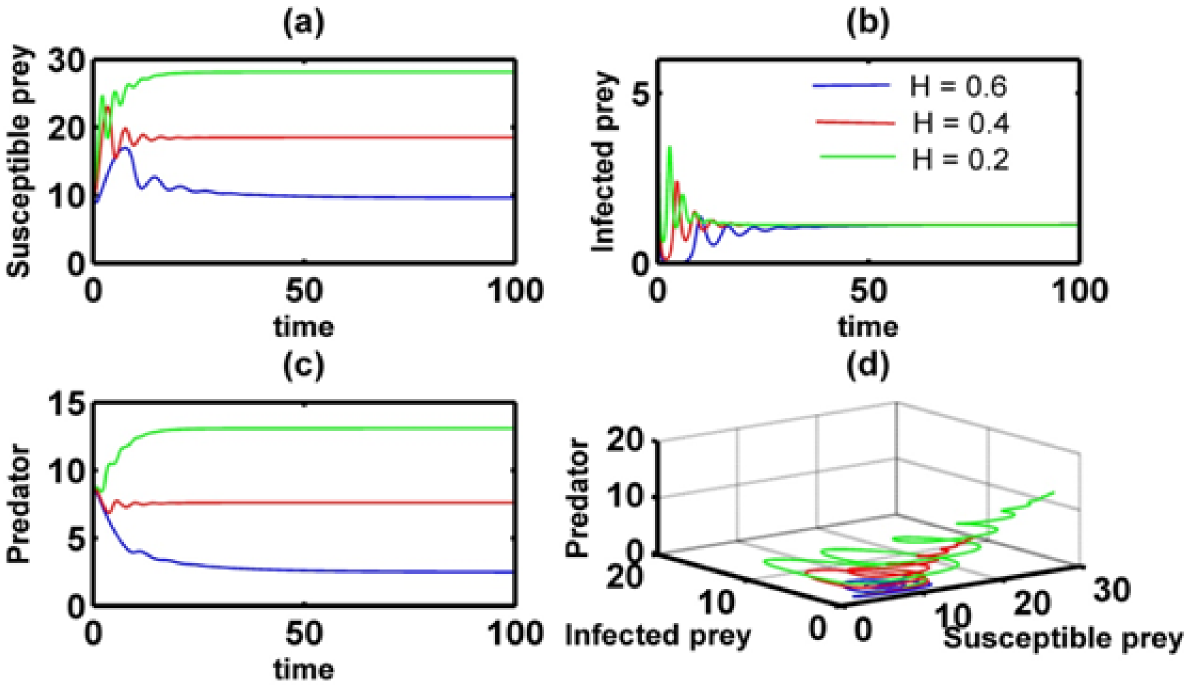

Fig. 10. Time series plot and phase portrait of the system (4) at $\sigma = 0.99$ for different values of H.

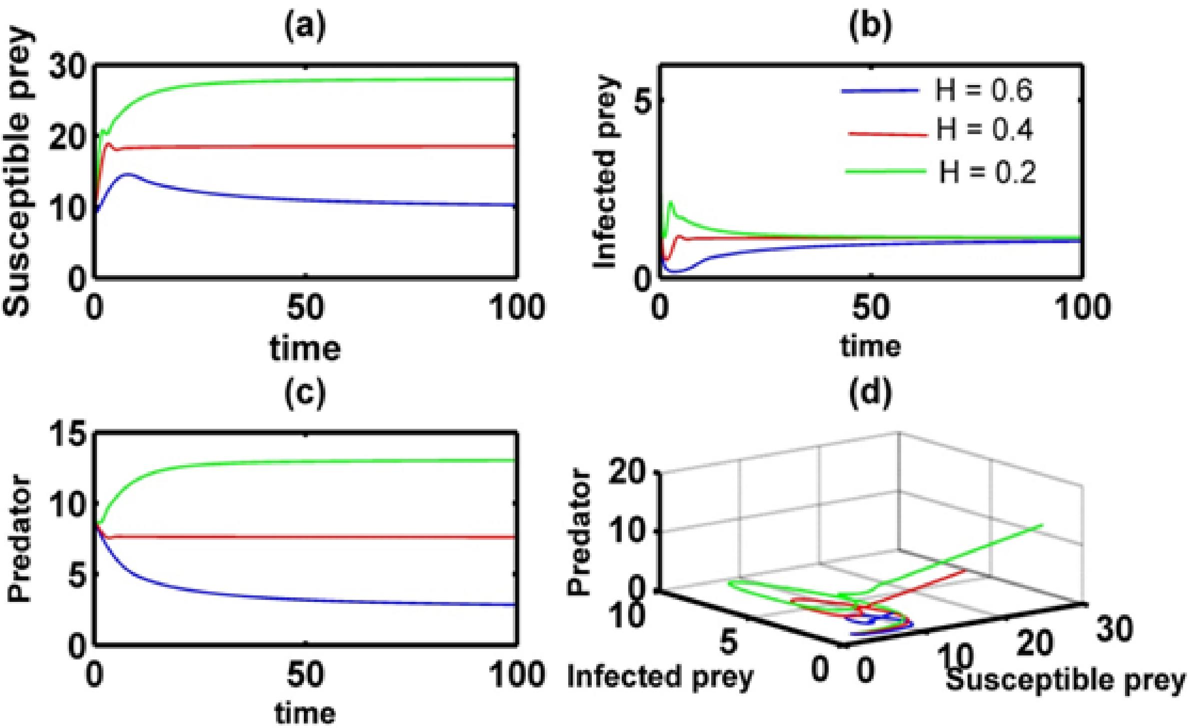

Fig. 11. Time series plot and phase portrait of the system (4) at $\sigma = 0.85$ for different values of H.

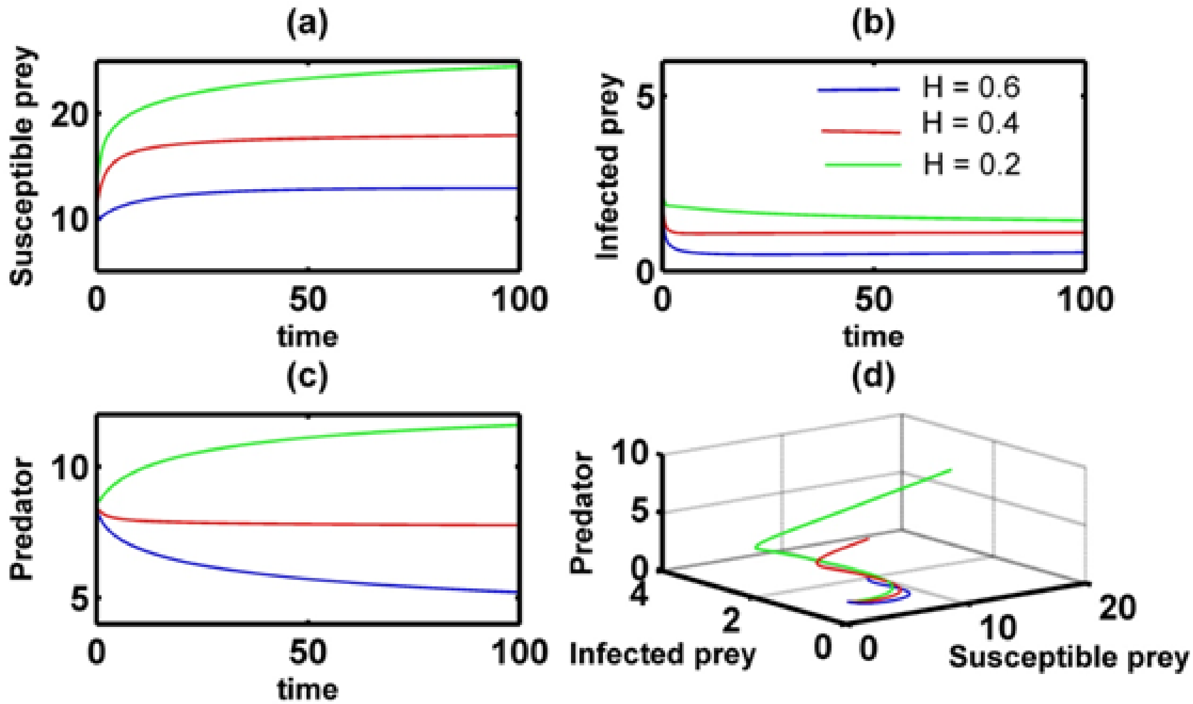

Fig. 12. Time series plot and phase portrait of the system (4) at $\sigma = 0.45$ for different values of H.

Conclusions

In this paper, an eco–epidemic prey–predator model with two stages of prey population and one stage of predator population is considered. As infection spreads through prey, the prey population is divided into two parts. Predator consumes susceptible prey and infected prey with Holling type-I and Holling type-II functional response respectively. In addition, we exploit the effect of harvesting to control the excessive spread of the infection. In addition, the fractional order derivative is utilized rather using ordinary differential equation. Therefore, the prey–predator ordinary differential equation system turns into a fractional order prey–predator system.

In the theoretical section, we investigate the existence and uniqueness of the solutions, non-negativity and boundedness which indicates that the solution of the system are non-negative and bounded uniformly if $E_2 < E_1$. The existence and local stability of five equilibrium points namely, the trivial equilibrium, the boundary or axial equilibrium, the infection free equilibrium, the predator free equilibrium and the interior equilibrium are investigated. We also examine the global stability condition at the interior equilibrium point of the fractional order prey-predator system. Numerical simulation is presented by the process based on the predictor-corrector PECE method of Adams-Bashforth-Moulton scheme. We use the solver function FDE12 to solve non-linear differential equation of fractional order (FDE). We see that the fractional order mathematical model can be useful to understand the system dynamics with working memory. From this study, we also observe that for the different values of the fractional derivative, the dynamics of the model such as equilibrium points, stability etc. can vary. These studies suggest that for increment of harvesting effort population of the species i.e., susceptible prey, infected prey and predator decreases. Therefore, to control the infection, selective harvesting is one of the biological and essential methods.

Due to lack of sufficient theoretical progress in fractional order calculus, it is very hard to find an exact solution of the fractional order mathematical model. Advanced numerical tools are necessary to simulate the fractional order model perfectly. Hopefully, we overcome these limitations in near future. As a future scope, one can explore the fractional order calculus in different imprecise environments, such as fuzzy environment, intuitionistic fuzzy environment, stochastic environment etc. In addition, this model can be extended to an eco-epidemic delay fractional order mathematical model.

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