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Growth estimates of entire functions under the flavour of slowly changing functions

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Abstract

Let t be a non-negative integer. In this paper we introduce the idea of generalized relative $L^*[t](\alpha,\beta)$-order and generalized relative $L^*[t](\alpha,\beta)$-type of an entire function with respect to another entire function and then we study some growth properties of entire functions on the basis of their generalized relative $L^*[t](\alpha,\beta)$- order and generalized relative $L^*[t](\alpha,\beta)$- type, where α, β are non negative continuous functions defined on (-∞,+∞). We give some examples which validates the theorems stated.

Keywords: Entire functions, growth, generalized relative $L^*[t](\alpha,\beta)$-order, generalized relative $L^*[t](\alpha,\beta)$-type, generalized relative $L^*[t](\alpha,\beta)$-weak type.

1. Introduction

We denote by $\mathbb{C}$, the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of $f=\sum_{n=0}^{\infty} a_n z^n$ on $|z|=r$ is defined as

$$M_f(r)=\max\{|f(z)|:|z|=r\}.$$

Moreover if f is non constant entire then $M_f(r)$ is also strictly increasing and continuous function of r. Therefore, its inverse

$$M_f^{-1}:(M_f(0),\infty)\to(0,\infty)$$

exists and is such that

$$\lim_{s\to\infty} M_f^{-1}(s)=\infty.$$

We use the standard notations and definitions of the theory of entire functions which are available in [13], [14] and therefore we do not explain those in details. Let us consider that the reader is familiar with the fundamental results of the Nevanlinna theory of meromorphic functions which are available in [7], [14].

For $x\in[0,\infty)$, we define iteration of logarithmic and exponential functions as

$$log^{[k]}x=\log(log^{[k-1]}x) \text{ for k=1,2,3,.....}$$

$$log^{[0]}x=x, log^{[-1]}x=expx$$

and

$$exp^{[k]}x=\exp(exp^{[k-1]}x) \text{ for } k=1,2,3,\ldots..$$

$$exp^{[0]}x=x, exp^{[-1]}x=logx.$$

However, let K be a class of continuous non negative function α on (-∞,+∞) such that $\alpha(x)=\alpha(x_0)$ for $x\le x_0$ with $\alpha(x)\uparrow+\infty$ as $x\to+\infty$. For any $\alpha\in K$, we say that $\alpha\in K_1^0$ if

$$\alpha((1+O(1))x)=(1+O(1))\,\alpha(x) \text{ as } x\to+\infty$$

and $\alpha\in K_2^0$ if

$$\alpha(\exp(1+O(1))x)=(1+O(1))\,\alpha(\exp(x)) \text{ as } x\to+\infty.$$

Finally, for any $\alpha\in K$, we also say that $\alpha\in K_1$ if

$$\alpha(cx)=(1+O(1))\,\alpha(x) \text{ as } x_0\le x\to+\infty$$

for each c ∈(0,+∞) ,

and $\alpha\in K_2$ if

$$\alpha(\exp(cx))=(1+O(1))\,\alpha(\exp(x)) \text{ as}$$

$x_0\le x\to+\infty$ for each c ∈(0,+∞).

Clearly

$$K_1\subset K_1^0,\ K_2\subset K_2^0 \text{ and } K_2\subset K_1.$$

Considering this, the value

$$\rho_{(\alpha,\beta)}=\limsup_{r\to\infty}\frac{\alpha(\log M_f(r))}{\beta(\log r)} \quad (\alpha\in K,\ \beta\in K)$$

and

$$\lambda_{(\alpha,\beta)}=\liminf_{r\to\infty}\frac{\alpha(\log M_f(r))}{\beta(\log r)} \quad (\alpha\in K,\ \beta\in K).$$

are respectively called generalized (α,β) order and generalized lower (α,β)- order of an entire function [10]. For the purpose of future applications, several authors rewrite the definition of generalized (α,β) order of entire and meromorphic function after giving a minor modification to the original definition. Again for α ∈K, β ∈K

$$\tau_{(\alpha,\beta)}[f]=\limsup_{r\to\infty}\frac{\exp\left(\alpha\left(M_f(r)\right)\right)}{exp(\beta(r))^{\lambda_{(\alpha,\beta)}}}$$

and

$$\bar{\tau}_{(\alpha,\beta)}[f]=\liminf_{r\to\infty}\frac{\exp\left(\alpha\left(M_f(r)\right)\right)}{exp\left(\beta(r)\right)^{\lambda_{(\alpha,\beta)}}},$$

are respectively called generalized (α,β)-upper weak type and generalized (α,β)-lower weak type of an entire function f [3]. where $(0<\lambda_{(\alpha,\beta)}[f]<\infty)$.

Let L≡L(r) be a positive continuous function increasing slowly that is L(ar)~L(r) as r→∞ for every positive constant a. Singh and Barker [11] defined it in the following way

A positive continuous function L(r) is called slowly changing function if, for ε(>0);

$$\frac{1}{k^{\varepsilon}}\le\frac{L(kr)}{L(r)}\le k^{\varepsilon} \text{ for } r\ge r(\varepsilon).$$

and uniformly for k(≥1).

If further, L(r) is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty}\frac{rL'(r)}{L(r)}=0$$

Somasundaram and Thamizharasi [12] introduced the notions of L-order and L-type for entire function. The more generalized concept for L-order and L-type for

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entire functions are L^* -order and L^* -type. During the past decades, several authors made closed investigations on the properties of entire functions on the basis of slowly changing functions in some different directions, so we get many important results from [4], [5], [6], [9].

Definition 1.1 : [7] The order and lower order of an entire function f are defined as

$$\rho_f = \lim_{r\to\infty} sup \frac{log^{[2]}M_f(r)}{logr}$$

and

$$\lambda_f = \lim_{r\to\infty} inf \frac{log^{[2]}M_f(r)}{logr}.$$

Definition 1.2 : Let α, β ∈K. Then we define generalized (α,β)- order denoted by $\rho^{[p]}_{(\alpha,\beta)}[f]$and generalized (α,β)-lower order denoted by $\lambda^{[p]}_{(\alpha,\beta)}[f]$ of an entire function f as

$$\rho^{[p]}_{(\alpha,\beta)}[f] = \lim_{r\to\infty} sup \frac{\alpha^{[p]}\left(M_f(r)\right)}{\beta(r)}$$

And

$$\lambda^{[p]}_{(\alpha,\beta)}[f] = \lim_{r\to\infty} sup \frac{\alpha^{[p]}\left(M_f(r)\right)}{\beta(r)}.$$

where p≥1.

Definition 1.3 : Let α, β ∈K, where K is defined earlier. Then we define generalized (α,β)-type denoted by $\sigma^{[p]}_{(\alpha,\beta)}[f]$ and generalized (α,β)-lower type denoted by $\overline{\sigma}^{[p]}_{(\alpha,\beta)}[f]$ of an entire function f having finite positive generalized (α,β)- order as 0<$\rho^{[p]}_{(\alpha,\beta)}[f] < \infty$ as

$$\sigma^{[p]}_{(\alpha,\beta)}[f] = \lim_{r\to\infty} sup \frac{\exp\left(\alpha^{[p]}\left(M_f(r)\right)\right)}{(exp(\beta(r)))^{\rho^{[p]}_{(\alpha,\beta)}[f]}},$$

and

$$\overline{\sigma}^{[p]}_{(\alpha,\beta)}[f] = \lim_{r\to\infty} inf \frac{\exp\left(\alpha^{[p]}\left(M_f(r)\right)\right)}{(exp(\beta(r)))^{\rho^{[p]}_{(\alpha,\beta)}[f]}},$$

where p≥1. It is obvious that $0 \le \overline{\sigma}^{[p]}_{(\alpha,\beta)}[f] \le \sigma^{[p]}_{(\alpha,\beta)}[f] \le \infty$.

Similarly, we can define generalized (α,β)-upper weak type denoted by $\tau^{[p]}_{(\alpha,\beta)}[f]$ and generalized (α,β)-lower weak type denoted by $\overline{\tau}^{[p]}_{(\alpha,\beta)}[f]$ of an entire function f having finite positive generalized (α,β)-lower order (0< $\lambda^{[p]}_{(\alpha,\beta)}[f]$<∞) as

$$\tau^{[p]}_{(\alpha,\beta)}[f] = \lim_{r\to\infty} sup \frac{\exp\left(\alpha^{[p]}\left(M_f(r)\right)\right)}{(exp(\beta(r)))^{\lambda^{[p]}_{(\alpha,\beta)}[f]}}$$

and

$$\overline{\tau}^{[p]}_{(\alpha,\beta)}[f] = \lim_{r\to\infty} inf \frac{\exp\left(\alpha^{[p]}\left(M_f(r)\right)\right)}{(exp(\beta(r)))^{\lambda^{[p]}_{(\alpha,\beta)}[f]}}$$

where p≥1. It is obvious that $0 \le \overline{\tau}^{[p]}_{(\alpha,\beta)}[f] \le \tau^{[p]}_{(\alpha,\beta)}[f] \le \infty$.

Definition 1.4 : Let f, g be two entire functions. Bernal [1], [2] initiated the definition of relative order $\rho_g(f)$ of f with respect to g which keep away from comparing growth just with exp z to find out order of entire functions as follows

$$\rho_g(f) = inf\{\mu > 0: M_f(r) < M_g(r^\mu)\}$$

$$= \lim_{r\to\infty} sup \frac{log M_g^{-1} M_f(r)}{logr}$$

Analogously, one may define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as

$$\lambda_g(f) = \lim_{r\to\infty} inf \frac{log M_g^{-1} M_f(r)}{logr}.$$

However, an entire function, for which order and lower order are the same, is said to be of regular growth. The function exp z is an example of regular growth of entire functions. Further the functions which are not of regular growth are said to be of irregular growth.

Definition 1.5 : Let f and g be any two entire functions with maximum modulus functions $M_f(r)$ and $M_g(r)$ respectively, then for any positive integer p, generalized relative (α,β)- order (respectively generalized relative (α,β)- lower order) of f with respect to g, denoted by $\rho^{[p]}_{(\alpha,\beta)}[f]_g$ (respectively $\lambda^{[p]}_{(\alpha,\beta)}[f]_g$) is defined as

$$\rho^{[p]}_{(\alpha,\beta)}[f]_g = \lim_{r\to\infty} sup \frac{\alpha^{[p]}\left(M_g^{-1} M_f(r)\right)}{\beta(r)}$$

and

$$\lambda^{[p]}_{(\alpha,\beta)}[f]_g = \lim_{r\to\infty} inf \frac{\alpha^{[p]}\left(M_g^{-1} M_f(r)\right)}{\beta(r)}$$

Definition 1.6 : Let α, β∈K and f and g be any two entire functions with maximum modulus functions $M_f(r)$ and $M_g(r)$ respectively, then for any positive integer p, we define generalized relative (α,β)- type denoted by $\sigma^{[p]}_{(\alpha,\beta)}[f]_g$ and generalized relative (α,β)- lower type denoted by $\overline{\sigma}^{[p]}_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to another entire function g having finite positive generalized relative (α,β) order (0< $\rho^{[p]}_{(\alpha,\beta)}[f]_g$ <∞)as

$$\sigma^{[p]}_{(\alpha,\beta)}[f]_g = \lim_{r\to\infty} sup \frac{exp\left(\alpha^{[p]}\left(M_g^{-1} M_f(r)\right)\right)}{(exp(\beta(r)))^{\rho^{[p]}_{(\alpha,\beta)}[f]_g}}$$

and

$$\overline{\sigma}^{[p]}_{(\alpha,\beta)}[f]_g = \lim_{r\to\infty} inf \frac{exp\left(\alpha^{[p]}\left(M_g^{-1} M_f(r)\right)\right)}{\left(exp\left(\beta(r)\right)\right)^{\rho^{[p]}_{(\alpha,\beta)}[f]_g}}.$$

It is obvious that $0 \le \overline{\sigma}^{[p]}_{(\alpha,\beta)}[f]_g \le \sigma^{[p]}_{(\alpha,\beta)}[f]_g \le \infty$.

Similarly, we can define generalized relative upper weak type (α,β) denoted by $\tau^{[p]}_{(\alpha,\beta)}[f]_g$ and generalized lower weak type (α,β) denoted by $\overline{\tau}^{[p]}_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to another entire function g having finite positive generalized relative lower order (α,β) ($0 < \lambda^{[p]}_{(\alpha,\beta)}[f]_g$ <∞) as

$$\tau^{[p]}_{(\alpha,\beta)}[f]_g = \lim_{r\to\infty} sup \frac{exp\left(\alpha^{[p]}\left(M_g^{-1} M_f(r)\right)\right)}{\left(exp\left(\beta(r)\right)\right)^{\lambda^{[p]}_{(\alpha,\beta)}[f]_g}}$$

and

$$\overline{\tau}_{(\alpha,\beta)}^{[p]}[f]_g = \lim_{r\to\infty} inf \frac{exp\left(\alpha^{[p]}\left(M_g^{-1}M_f\,(r)\right)\right)}{(exp(\beta(r)))^{\lambda_{(\alpha,\beta)}^{[p]}[f]_g}},$$

where p≥1. It is obvious that $0 \le \overline{\tau}_{(\alpha,\beta)}^{[p]}[f]_g \le \tau_{(\alpha,\beta)}^{[p]}[f]_g \le \infty$.

Definition 1.7 : Let α, β ∈K. Then we define L^*(α,β)-order and L^*(α,β)-lower order as

$$\rho_{(\alpha,\beta)}^{L^*}[f] = \lim_{r\to\infty} sup \frac{\alpha\left(M_f\,(r)\right)}{\beta\left(re^{L(r)}\right)}$$

and

$$\lambda_{(\alpha,\beta)}^{L^*}[f] = \lim_{r\to\infty} inf \frac{\alpha\left(M_f\,(r)\right)}{\beta\left(re^{L(r)}\right)}$$

Similarly, we can define $L^*[t]$(α,β)-order and $L^*[t]$(α,β)-lower order as

$$\rho_{(\alpha,\beta)}^{L^*[t]}[f] = \lim_{r\to\infty} sup \frac{\alpha\left(M_f\,(r)\right)}{\beta\left(rexp^{[t]}L(r)\right)}$$

and

$$\lambda_{(\alpha,\beta)}^{L^*[t]}[f] = \lim_{r\to\infty} inf \frac{\alpha\left(M_f\,(r)\right)}{\beta\left(rexp^{[t]}L(r)\right)}.$$

where t is any non negative integer.

Definition 1.8 : Let α,β∈K. Then we define relative L^*(α,β)-order and relative L^*(α,β)-lower order of an entire function f with respect to another entire function g as,

$$\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g = \lim_{r\to\infty} sup \frac{\alpha\left(M_g^{-1}M_f\,(r)\right)}{\beta\left(rexp^{[t]}L(r)\right)}$$

and

$$\lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g = \lim_{r\to\infty} inf \frac{\alpha\left(M_g^{-1}M_f\,(r)\right)}{\beta\left(rexp^{[t]}L(r)\right)},$$

where t is any non negative integer.

Definition 1.9 : Let α, β∈K and f and g be any two entire functions with maximum modulus functions $M_f\,(r)$ and $M_g\,(r)$ respectively, then for any positive integer p, we define generalized relative $L^*[t]$(α,β)-type denoted by $\sigma_{(\alpha,\beta)}^{L^*[t]}[f]_g$ and generalized relative $L^*[t]$(α,β)-lower type denoted by $\overline{\sigma}_{(\alpha,\beta)}^{L^*[t]}[f]_g$of an entire function f with respect to another entire function g having finite positive generalized relative (α,β)- order ($0<\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g < \infty$) as

$$\sigma_{(\alpha,\beta)}^{L^*[t]}[f]_g = \lim_{r\to\infty} sup \frac{exp\left(\alpha\left(M_g^{-1}M_f\,(r)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g}}$$

and

$$\overline{\sigma}_{(\alpha,\beta)}^{L^*[t]}[f]_g = \lim_{r\to\infty} inf \frac{exp\left(\alpha\left(M_g^{-1}M_f\,(r)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g}},$$

where t is any non negative integer. It is obvious that $0 \le \sigma_{(\alpha,\beta)}^{L^*[t]}[f]_g \le \overline{\sigma}_{(\alpha,\beta)}^{L^*[t]}[f]_g \le \infty$.

Similarly, we define generalized relative $L^*[t]$(α,β)-upper weak type denoted by $\tau_{(\alpha,\beta)}^{L^*[t]}[f]_g$and generalized relative $L^*[t]$(α,β)-lower weak type denoted by $\overline{\tau}_{(\alpha,\beta)}^{L^*[t]}[f]_g$of an entire function f with respect to another entire function g having finite positive generalized relative (α,β)- lower order ($0<\lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g <\infty$) as

$$\tau_{(\alpha,\beta)}^{L^*[t]}[f]_g = \lim_{r\to\infty} sup \frac{exp\left(\alpha\left(M_g^{-1}M_f\,(r)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g}}$$

and

$$\overline{\tau}_{(\alpha,\beta)}^{L^*[t]}[f]_g = \lim_{r\to\infty} inf \frac{exp\left(\alpha\left(M_g^{-1}M_f\,(r)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g}}.$$

It is obvious that $0 \le \overline{\tau}_{(\alpha,\beta)}^{L^*[t]}[f]_g \le \tau_{(\alpha,\beta)}^{L^*[t]}[f]_g \le \infty$.

Definition 1.10: Let α, β∈ K, where K is defined earlier and f and g be any two entire functions with maximum modulus functions
where p≥1 and t is any non-negative integer.
and f be an entire function with maximum modulus function $M_f\,(r)$ respectively, then for any positive integer p, we define generalized $L^*[t]$(α,β) -type denoted by $\sigma_{(\alpha,\beta)}^{[p]L^*[t]}[f]$ and generalized $L^*[t]$(α,β) -lower type denoted by $\overline{\sigma}_{(\alpha,\beta)}^{[p]L^*[t]}[f]$of an entire function f having finite positive generalized (α,β)- order as

$$\sigma_{(\alpha,\beta)}^{[p]L^*[t]}[f] = \lim_{r\to\infty} sup \frac{exp\left(\alpha^{[p]}\left(M_f\,(r)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\alpha,\beta)}^{[p]L^*[t]}[f]}}$$

and

$$\overline{\sigma}_{(\alpha,\beta)}^{[p]L^*[t]}[f]_g = \lim_{r\to\infty} inf \frac{exp\left(\alpha^{[p]}\left(M_f\,(r)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\alpha,\beta)}^{L^*[t]}[f]}},$$

where t is any non negative integer. It is obvious that $0 \le \overline{\sigma}_{(\alpha,\beta)}^{[p]L^*[t]}[f] \le \sigma_{(\alpha,\beta)}^{[p]L^*[t]}[f] \le \infty$.

In the analogous way, we can define generalized $L^*[t]$(α,β)- upper weak type denoted by $\tau_{(\alpha,\beta)}^{[p]L^*[t]}[f]$ and generalized $L^*[t]$(α,β) -lower weak type denoted by $\overline{\tau}_{(\alpha,\beta)}^{[p]L^*[t]}[f]_g]$ of an entire function f having finite positive generalized relative (α,β)- lower order as

$$\tau_{(\alpha,\beta)}^{[p]L^*[t]}[f] = \lim_{r\to\infty} sup \frac{exp\left(\alpha^{[p]}\left(M_f\,(r)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\lambda_{(\alpha,\beta)}^{[p]L^*[t]}[f]}}$$

and

$$\overline{\tau}_{(\alpha,\beta)}^{[p]L^*[t]}[f] = \lim_{r\to\infty} inf \frac{exp\left(\alpha^{[p]}\left(M_f\,(r)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\lambda_{(\alpha,\beta)}^{[p]L^*[t]}[f]}},$$

where p≥1 and t is any non-negative integer. It is obvious that $0 \le \overline{\tau}_{(\alpha,\beta)}^{[p]L^*[t]}[f] \le \tau_{(\alpha,\beta)}^{[p]L^*[t]}[f] \le \infty$.

2. Main Results

Theorem 2.1: Let $f$and $g$be any two entire functions such that

$$0 < \lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f] \le \rho_{(\gamma,\beta)}^{[p]L^*[t]}[f] < \infty$$

and

$$0<\lambda_{(\gamma,\alpha)}^{[p]}[g]\le\rho_{(\gamma,\alpha)}^{[p]}[g]<\infty,$$

where $p$are any positive integer. Then

$$\left[\frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}\right]\le\lambda_{(\gamma,\beta)}^{L^*[t]}[f]_g$$

$$\le\min\left\{\left[\frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}\right],\left[\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}\right]\right\}$$

$$\le\max\left\{\left[\frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}\right],\left[\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}\right]\right\}$$

$$\le\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g\le\left[\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}\right]$$

Proof: From the definition of $\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]$ and $\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]$ we have for all sufficiently large values of r that

$$M_f(r)\le\gamma^{[-p]}\left\{\left(\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]+\epsilon\right)\beta\left(rexp^{[t]}L(r)\right)\right\}\,(1)$$

$$M_f(r)\ge\gamma^{[-p]}\left\{\left(\lambda_{(\alpha,\beta)}^{[p]L^*[t]}[f]-\epsilon\right)\beta\left(rexp^{[t]}L(r)\right)\right\}\,(2)$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r)\ge\gamma^{[-p]}\left\{\left(\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]-\epsilon\right)\beta\left(rexp^{[t]}L(r)\right)\right\}\,(3)$$

$$M_f(r)\le\gamma^{[-p]}\left\{\left(\lambda_{(\alpha,\beta)}^{[p]L^*[t]}[f]+\epsilon\right)\beta\left(rexp^{[t]}L(r)\right)\right\}\,(4)$$

Similarly from the definition of $\rho_{(\gamma,\alpha)}^{[p]}[g]$ and $\lambda_{(\gamma,\alpha)}^{[p]}[g]$, it follows for all sufficiently large values of r that

$$M_g(r)\le\gamma^{[-p]}\left\{\left(\rho_{(\gamma,\alpha)}^{[p]}[g]+\epsilon\right)\alpha(r)\right\}$$

$$i.e.,\qquad(r)\le M_g^{-1}\left[\gamma^{[-p]}\left\{\left(\rho_{(\gamma,\alpha)}^{[p]}[g]+\epsilon\right)\alpha(r)\right\}\right],$$

$i.e.,$

$$M_g^{-1}\ge\alpha^{-1}\left[\frac{\gamma^{[p]}(r)}{\left(\rho_{(\gamma,\alpha)}^{[p]}[g]+\epsilon\right)}\right],\qquad(5)$$

and

$$M_g(r)\ge\gamma^{[-p]}\left\{\left(\lambda_{(\gamma,\alpha)}^{[p]}[g]-\epsilon\right)\alpha(r)\right\}$$

$i.e.,$

$$M_g^{-1}\le\alpha^{-1}\left[\frac{\gamma^{[p]}(r)}{\left(\lambda_{(\gamma,\alpha)}^{[p]}[g]-\epsilon\right)}\right],\qquad(6)$$

also from the definition of $\rho_{(\gamma,\alpha)}^{[p]}[g]$and $\lambda_{(\gamma,\alpha)}^{[p]}[g]$, for a sequence of values of r tending to ∞, we obtain that

$$M_g(r)\ge\gamma^{[-p]}\left\{\left(\rho_{(\gamma,\alpha)}^{[p]}[g]-\epsilon\right)\alpha(r)\right\},$$

$$i.e.,\qquad M_g^{-1}\le\alpha^{-1}\left[\frac{\gamma^{[p]}(r)}{\left(\rho_{(\gamma,\alpha)}^{[p]}[g]-\epsilon\right)}\right]\,(7)$$

and

$$M_g(r)\le\gamma^{[-p]}\left\{\left(\lambda_{(\gamma,\alpha)}^{[p]}[g]+\epsilon\right)\alpha(r)\right\},$$

$$i.e.,\qquad M_g^{-1}\ge\alpha^{-1}\left[\frac{\gamma^{[p]}(r)}{\left(\lambda_{(\gamma,\alpha)}^{[p]}[g]+\epsilon\right)}\right].\,(8)$$

Now from Equation (3) and in view of Equation (5), we get for a sequence of values of r tending to infinity we get that

$$\alpha\left(M_g^{-1}M_f(r)\right)\ge\alpha\left[M_g^{-1}\left(\gamma^{[-p]}\left\{\left(\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]-\epsilon\right)\beta\left(rexp^{[t]}L(r)\right)\right\}\right)\right]$$

$i.e.,$

$$\alpha\left(M_g^{-1}M_f(r)\right)\ge\alpha\left[\alpha^{-1}\left(\frac{\gamma^{[p]}\left(\gamma^{[-p]}\left\{\left(\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]-\epsilon\right)\beta\left(rexp^{[t]}L(r)\right)\right\}\right)}{\left(\rho_{(\gamma,\alpha)}^{[p]}[g]+\epsilon\right)}\right)\right],$$

$$i.e.,\qquad\alpha\left(M_g^{-1}M_f(r)\right)\ge\left[\frac{\left(\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]-\epsilon\right)\beta\left(rexp^{[t]}L(r)\right)}{\left(\rho_{(\gamma,\alpha)}^{[p]}[g]+\epsilon\right)}\right],$$

$$i.e.,\frac{\alpha\left(M_g^{-1}M_f(r)\right)}{\beta\left(rexp^{[t]}L(r)\right)}\ge\frac{\left(\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]-\epsilon\right)}{\left(\rho_{(\gamma,\alpha)}^{[p]}[g]+\epsilon\right)}.$$

As $\epsilon\ (>0)$is arbitrary, it follows that

$$\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g\ge\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}\,(9)$$

Analogously from Equation (2) and in view of Equation (8), for a sequence of values of r tending to infinity we get that

$$\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g\ge\frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}\,(10)$$

and from Equation (2) and in view of Equation (5), with the same reasoning we get that

$$\lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g\ge\frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}\,(11)$$

Again in view of Equation (6),we have from Equation (1) for all sufficiently large values of r that

$$\alpha\left(M_g^{-1}M_f(r)\right)\le\alpha\left[M_g^{-1}\left(\gamma^{[-p]}\left\{\left(\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]+\epsilon\right)\beta\left(rexp^{[t]}L(r)\right)\right\}\right)\right],$$

$i.e.,$

$$\alpha\left(M_g^{-1}M_f(r)\right)\le\alpha\left[\alpha^{-1}\left(\frac{\gamma^{[p]}\left(\gamma^{[-p]}\left\{\left(\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]+\epsilon\right)\beta\left(rexp^{[t]}L(r)\right)\right\}\right)}{\left(\lambda_{(\gamma,\alpha)}^{[p]}[g]-\epsilon\right)}\right)\right],$$

$$i.e.,\frac{\alpha\left(M_g^{-1}M_f(r)\right)}{\beta\left(rexp^{[t]}L(r)\right)}\le\frac{\left(\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]+\epsilon\right)}{\left(\lambda_{(\gamma,\alpha)}^{[p]}[g]-\epsilon\right)}.$$

As $\epsilon\ (>0)$is arbitrary, it follows that

$$\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g\le\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}\,(12)$$

Also in view of Equation (7), we get from Equation (1) that

$$\lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g\le\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}\,(13)$$

Similarly from Equation (4) and in view of Equation (6), it follows that

$$\lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g\le\frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}\,(14)$$

Then the theorem follows from Equation (9), (10), (11), (12), (13) and (14).

Example
Let $\alpha(r) = \beta(r) = \gamma(r) = \sin r$, $f(z) = z^3$, $g(z) = z$, $L(r) = log^{[2]}(r^2)$, $t = 2$ and $p = 1$ in $|z| \leq r$. Then

$$\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f] = \liminf_{r\to\infty} \frac{\sin r^3}{\sin(rexp^{[2]}log^{[2]}(r^2))} = 1$$

Similarly we obtained that $\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f] = 1$.
Again

$$\lambda_{(\gamma,\alpha)}^{[p]}[g] = \liminf_{r\to\infty} \frac{\sin r}{\sin r} = 1$$

and similarly we get, $\rho_{(\gamma,\alpha)}^{[p]}[g] = 1$.
Again

$$\lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g = \liminf_{r\to\infty} \frac{\sin r^3}{\sin(rexp^{[2]}log^{[2]}(r^2))} = 1$$

and also $\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g = 1$.
Hence,

$$\frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]} = 1,$$

$$min\left\{\left[\frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}\right], \left[\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}\right]\right\} = 1,$$

$$max\left\{\left[\frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}\right], \left[\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}\right]\right\} = 1,$$

and

$$\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]} = 1,$$

which validates Theorem 2.1.
Remark: From the conclusion of the above result one may write

$$\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g = \frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}$$

and

$$\lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g = \frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}$$

when $\lambda_{(\gamma,\alpha)}^{[p]}[g] = \rho_{(\gamma,\alpha)}^{[p]}[g]$. Similarly

$$\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g = \frac{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}$$

and

$$\lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g = \frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\rho_{(\gamma,\alpha)}^{[p]}[g]}$$

when $\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g = \lambda_{(\alpha,\beta)}^{L^*[t]}[f]_g$.

<u>Theorem 2.2</u>: Let f and g be any two entire functions such that

$$\mathbf{0} < \rho_{(\gamma,\beta)}^{[p]L^*[t]}[f] < \infty$$

and

$$0 < \lambda_{(\gamma,\alpha)}^{[p]}[g] \leq \rho_{(\gamma,\alpha)}^{[p]}[g] < \infty,$$

where p are any positive integer. Then

$$max\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\overline{\tau}_{(\gamma,\alpha)}^{[p]}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}, \left(\frac{\sigma_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\tau_{(\gamma,\alpha)}^{[p]}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}\right\}$$

$$\leq \sigma_{(\gamma,\beta)}^{L^*[t]}[f]_g \leq \left(\frac{\sigma_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\overline{\sigma}_{(\gamma,\alpha)}^{[p]}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}^{[p]}[g]}}.$$

Proof: Let us consider that ε(>0) is arbitrary. Now from the definitions of $\sigma_{(\gamma,\beta)}^{[p]L^*[t]}[f]$ *and* $\overline{\sigma}_{(\gamma,\beta)}^{[p]L^*[t]}[f]$, we have for all sufficiently large values of r that

$$M_f(r) \leq \gamma^{[-p]}\left[log\left(\left(\sigma_{(\gamma,\beta)}^{[p]L^*[t]}[f] + \varepsilon\right)\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}\right)\right] \quad (15)$$

and

$$M_f(r) \geq \gamma^{[-p]}\left[log\left(\left(\overline{\sigma}_{(\gamma,\beta)}^{[p]L^*[t]}[f] - \varepsilon\right)\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}\right)\right] \quad (16)$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq \gamma^{[-p]}\left[log\left(\left(\sigma_{(\gamma,\beta)}^{[p]L^*[t]}[f] - \varepsilon\right)\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}\right)\right] \quad (17)$$

and

$$M_f(r) \leq \gamma^{[-p]}\left[log\left(\left(\overline{\sigma}_{(\gamma,\beta)}^{[p]L^*[t]}[f] + \varepsilon\right)\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}\right)\right]. \quad (18)$$

Similarly, from definitions of $\sigma_{(\gamma,\alpha)}^{[p]}[g]$ and $\overline{\sigma}_{(\gamma,\alpha)}^{[p]}[g]$, it follows for all sufficiently large values of r that

$$M_g(r) \leq \gamma^{[-p]}\left[log\left(\left(\sigma_{(\gamma,\alpha)}^{[p]}[g] + \varepsilon\right)\left(exp(\alpha(r))\right)^{\rho_{(\gamma,\alpha)}^{[p]}[g]}\right)\right]$$

i.e.,

$$M_g^{-1}(r) \geq \alpha^{-1}\left(log\left(\frac{exp(\gamma^p(r))}{\left(\sigma_{(\gamma,\alpha)}^{[p]}[g]+\varepsilon\right)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}^{[p]}[g]}}\right) \quad (19)$$

and

$$M_g^{-1}(r) \leq \alpha^{-1}\left(log\left(\frac{exp(\gamma^p(r))}{\left(\overline{\sigma}_{(\gamma,\alpha)}^{[p]}[g]-\varepsilon\right)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}^{[p]}[g]}}\right). \quad (20)$$

Also for a sequence of values of r tending to infinity, we obtain that

$$M_g^{-1}(r) \leq \alpha^{-1}\left(log\left(\frac{exp(\gamma^p(r))}{\left(\sigma_{(\gamma,\alpha)}^{[p]}[g]-\varepsilon\right)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}^{[p]}[g]}}\right) \quad (21)$$

and

$$M_g^{-1}(r) \geq \alpha^{-1}\left(log\left(\frac{exp(\gamma^p(r))}{\left(\overline{\sigma}_{(\gamma,\alpha)}^{[p]}[g]+\varepsilon\right)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}^{[p]}[g]}}\right). \quad (22)$$

Further from the definitions of $\tau_{(\gamma,\beta)}^{[p]L^*[t]}[f]$ and $\overline{\tau}_{(\gamma,\beta)}^{[p]L^*[t]}[f]$ it follows that, for all sufficiently large values of r that

$$M_f(r) \leq \gamma^{[-p]}\left[log\left(\left(\tau_{(\gamma,\beta)}^{[p]L^*[t]}[f]+\varepsilon\right)\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}\right)\right] \quad (23)$$

and

$$M_f(r) \geq \gamma^{[-p]}\left[log\left(\left(\overline{\tau}_{(\gamma,\beta)}^{[p]L^*[t]}[f]-\varepsilon\right)\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}\right)\right]. \quad (24)$$

Also, for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq \gamma^{[-p]}\left[log\left(\left(\tau_{(\gamma,\beta)}^{[p]L^*[t]}[f]-\varepsilon\right)\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}\right)\right] \quad (25)$$

and

$$M_f(r) \leq \gamma^{[-p]}\left[log\left(\left(\overline{\tau}_{(\gamma,\beta)}^{[p]L^*[t]}[f]+\varepsilon\right)\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\lambda_{(\gamma,\beta)}^{[p]L^*[t]}[f]}\right)\right]. \quad (26)$$

Similarly from the definitions of $\tau_{(\gamma,\alpha)}^{[p]}[g]$ and $\overline{\tau}_{(\gamma,\alpha)}^{[p]}[g]$, it follows for all sufficiently large values of r that

$$M_g^{-1}(r) \geq \alpha^{-1}\left(log\left(\frac{exp(\gamma^p(r))}{\left(\tau_{(\gamma,\alpha)}^{[p]}[g]+\varepsilon\right)}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}\right) \quad (27)$$

and

$$M_g^{-1}(r) \leq \alpha^{-1}\left(log\left(\frac{exp(\gamma^p(r))}{\left(\overline{\tau}_{(\gamma,\alpha)}^{[p]}[g]-\varepsilon\right)}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}\right). \quad (28)$$

Also for a sequence of values of r tending to infinity we obtain that

$$M_g^{-1}(r) \leq \alpha^{-1}\left(log\left(\frac{exp(\gamma^p(r))}{\left(\tau_{(\gamma,\alpha)}^{[p]}[g]-\varepsilon\right)}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}\right) \quad (29)$$

and

$$M_g^{-1}(r) \geq \alpha^{-1}\left(log\left(\frac{exp(\gamma^p(r))}{\left(\overline{\tau}_{(\gamma,\alpha)}^{[p]}[g]+\varepsilon\right)}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}\right). \quad (30)$$

Now from Equation (17) and in view of Equation (27), we get for a sequence of values of r tending to infinity that

$$exp\left(\alpha\left(M_g^{-1}\left(M_f(r)\right)\right)\right) \geq \left[\frac{\left(\sigma_{(\gamma,\beta)}^{[p]L^*[t]}[f]-\varepsilon\right)\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}}{\left(\tau_{(\gamma,\alpha)}^{[p]}[g]+\varepsilon\right)}\right]^{\frac{1}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}$$

i.e.,

$$\frac{exp\left(\alpha\left(M_g^{-1}\left(M_f(r)\right)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}} \geq \left(\frac{\left(\sigma_{(\gamma,\beta)}^{[p]L^*[t]}[f]-\varepsilon\right)}{\left(\tau_{(\gamma,\alpha)}^{[p]}[g]+\varepsilon\right)}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}.$$

Since in view of Theorem 2.1

$$\frac{\rho_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\lambda_{(\gamma,\alpha)}^{[p]}[g]} \geq \rho_{(\alpha,\beta)}^{L^*[t]}[f]_g$$

for p≥1, so we get that

$$\frac{exp\left(\alpha\left(M_g^{-1}\left(M_f(r)\right)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g}} \geq \left(\frac{\sigma_{(\gamma,\beta)}^{[p]L^*[t]}[f]-\varepsilon}{\tau_{(\gamma,\alpha)}^{[p]}[g]+\varepsilon}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}.$$

As ε(>0) is arbitrary, therefore it follows from above that

$$\limsup_{r\to\infty}\frac{exp\left(\alpha\left(M_g^{-1}\left(M_f(r)\right)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho_{(\alpha,\beta)}^{L^*[t]}[f]_g}} \geq \left(\frac{\sigma_{(\gamma,\beta)}^{[p]L^*[t]}[f]}{\tau_{(\gamma,\alpha)}^{[p]}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}^{[p]}[g]}}.$$

So,

$$\sigma^{L^*[t]}_{(\alpha,\beta)}[f]_g \geq \left(\frac{\sigma^{[p]L^*[t]}_{(\gamma,\beta)}[f]}{\tau^{[p]}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda^{[p]}_{(\gamma,\alpha)}[g]}}. \qquad (31)$$

Analogously from Equation (16) and Equation (30) and in view of

$$\frac{\rho^{[p]L^*[t]}_{(\gamma,\beta)}[f]}{\lambda^{[p]}_{(\gamma,\alpha)}[g]} \geq \rho^{L^*[t]}_{(\alpha,\beta)}[f]_g$$

for p≥1, of Theorem 2.1 we get that

$$\sigma^{L^*[t]}_{(\alpha,\beta)}[f]_g \geq \left(\frac{\bar{\sigma}^{[p]L^*[t]}_{(\gamma,\beta)}[f]}{\bar{\tau}^{[p]}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda^{[p]}_{(\gamma,\alpha)}[g]}}. \qquad (32)$$

Again in view of Equation (20) we have from Equation (15) for all sufficiently large values of r that

$$\left(\alpha\left(M_g{}^{-1}\left(M_f(r)\right)\right)\right)$$

$$\leq \left[\frac{\left(\sigma^{[p]L^*[t]}_{(\gamma,\beta)}[f]+\varepsilon\right)\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho^{[p]L^*[t]}_{(\gamma,\beta)}[f]}}{\left(\bar{\sigma}^{[p]}_{(\gamma,\alpha)}[g]+\varepsilon\right)}\right]^{\frac{1}{\rho^{[p]}_{(\gamma,\alpha)}[g]}}$$

i.e.,

$$\frac{exp\left(\alpha\left(M_g{}^{-1}\left(M_f(r)\right)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\frac{\rho^{[p]L^*[t]}_{(\gamma,\beta)}[f]}{\rho^{[p]}_{(\gamma,\alpha)}[g]}}} \leq \left(\frac{\left(\sigma^{[p]L^*[t]}_{(\gamma,\beta)}[f]+\varepsilon\right)}{\left(\bar{\sigma}^{[p]}_{(\gamma,\alpha)}[g]-\varepsilon\right)}\right)^{\frac{1}{\rho^{[p]}_{(\gamma,\alpha)}[g]}}.$$

Since in view of Theorem 2.1 it follows that

$$\frac{\rho^{[p]L^*[t]}_{(\gamma,\beta)}[f]}{\rho^{[p]}_{(\gamma,\alpha)}[g]} \leq \rho^{L^*[t]}_{(\alpha,\beta)}[f]_g$$

for p≥1, so we get

$$\frac{exp\left(\alpha\left(M_g{}^{-1}\left(M_f(r)\right)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho^{L^*[t]}_{(\alpha,\beta)}[f]_g}} \leq \left(\frac{\left(\sigma^{[p]L^*[t]}_{(\gamma,\beta)}[f]+\varepsilon\right)}{\left(\bar{\sigma}^{[p]}_{(\gamma,\alpha)}[g]-\varepsilon\right)}\right)^{\frac{1}{\rho^{[p]}_{(\gamma,\alpha)}[g]}}.$$

Since ε(>0) is arbitrary, therefore it follows from above that

$$\lim_{r\to\infty} sup \frac{exp\left(\alpha\left(M_g{}^{-1}\left(M_f(r)\right)\right)\right)}{\left(exp\left(\beta\left(rexp^{[t]}L(r)\right)\right)\right)^{\rho^{L^*[t]}_{(\alpha,\beta)}[f]_g}}$$

$$\leq \left(\frac{\sigma^{[p]L^*[t]}_{(\gamma,\beta)}[f]}{\bar{\sigma}^{[p]}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho^{[p]}_{(\gamma,\alpha)}[g]}}$$

i.e.,

$$\sigma^{L^*[t]}_{(\alpha,\beta)}[f]_g \leq \left(\frac{\sigma^{[p]L^*[t]}_{(\gamma,\beta)}[f]}{\bar{\sigma}^{[p]}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho^{[p]}_{(\gamma,\alpha)}[g]}}. \qquad (33)$$

Hence the theorem follows from Equation (31),(32), (33).

Example: Let f(z)=exp(2z+z³)², g(z)=exp z, p=1, t=1, α(r)=β(r)=log r, γ=$log^{[2]}r$ and L(r)=log(r²+2), then we get

$$\rho^{[p]L^*[t]}_{(\gamma,\beta)}[f] = \lambda^{[p]L^*[t]}_{(\gamma,\beta)}[f] = 2,$$

$$\lambda^{[p]}_{(\gamma,\alpha)}[g] = \rho^{[p]}_{(\gamma,\alpha)}[g] = 1.$$

And

$$\sigma^{[p]L^*[t]}_{(\gamma,\beta)}[f]=\bar{\sigma}^{[p]L^*[t]}_{(\gamma,\beta)}[f] = 1$$

Also,

$$\bar{\sigma}^{[p]}_{(\gamma,\alpha)}[g] = \tau^{[p]}_{(\gamma,\alpha)}[g] = \sigma^{[p]}_{(\gamma,\alpha)}[g] = 1$$

and

$$\rho^{L^*[t]}_{(\alpha,\beta)}[f]_g = \sigma^{L^*[t]}_{(\alpha,\beta)}[f]_g = 1,$$

So,

$$max\left\{\left(\frac{\bar{\sigma}^{[p]L^*[t]}_{(\gamma,\beta)}[f]}{\bar{\tau}^{[p]}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda^{[p]}_{(\gamma,\alpha)}[g]}}, \left(\frac{\sigma^{[p]L^*[t]}_{(\gamma,\beta)}[f]}{\tau^{[p]}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda^{[p]}_{(\gamma,\alpha)}[g]}}\right\} = 1$$

and

$$\left(\frac{\sigma^{[p]L^*[t]}_{(\gamma,\beta)}[f]}{\bar{\sigma}^{[p]}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho^{[p]}_{(\gamma,\alpha)}[g]}} = 1.$$

which validates Theorem 2.2.

Conclusion and future prospect:

After introducing the idea of generalized relative $L^{*[t]}(\alpha,\beta)$ order and generalized relative $L^{*[t]}(\alpha,\beta)$ lower order of an entire function of complex variable with respect to an entire function, where α,β are non negative continuous functions defined on (-∞,+∞), here in this paper we study some growth properties of entire functions. This assumption is also used to modify the idea of generalized relative order (lower order) (α,β) and generalized relative type (lower type) (α,β) of an entire function as well as meromorphic function by using non-decreasing unbounded function Ψ, where Ψ:[0,∞)→(0,∞) satisfying the following two conditions:

$$\lim_{r\to\infty}\frac{log^{[p]} r}{log^{[q]}\Psi(r)} = 0$$

and

$$\lim_{r\to\infty}\frac{log^{[p]}\Psi(ar)}{log^{[q]}\Psi(r)} = 1$$

Taking this modification we derive some results which will no doubt inspire the future researcher to derive some growth properties of entire and meromorphic functions of n complex variables.

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