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On the Distribution of Poles of Meromorphic Functions in the Light of Slowly Changing Functions

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Abstract

A meromorphic function on $D \subseteq \mathbb{C}$ is a ratio of analytic functions with denominator identically non zero on D. Poles of such functions arise from zeros in the denominator where the numerator remains non-zero. Determining all poles is a complex task, thus identifying a potential pole region becomes essential. This research aims to establish a pole region for selected meromorphic functions, supported by examples and accompanying figures to validate the findings.

Keywords: Meromorphic function, pole, L-order, L^*-order.

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1. Introduction, Definitions and Notations

In 1816, Gauss initially presented a fundamental result concerning the location of zeros of polynomials [7]. Consequently, numerous scholarly articles exploring this topic have been published in the academic literature (cf. [1], [2], [3], [7], [8] & [11]). However, it's important to emphasize that more results of a similar nature for the location of poles of meromorphic functions are not available, except for a limited partial reflection observed in [5].

A meromorphic function $f(z)$ analytic within the annular region defined by $r_1 < |z| < r_2$ can be expressed as $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ where $a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi^{n+1}} d\xi, n \in \mathbb{Z}$ with $\Gamma = \{\xi : |\xi| = r\}$ and $r_1 < r < r_2$.

The primary objective of this paper is to introduce a region that encompasses the poles of a meromorphic function under various conditions, utilizing the coefficients a_n.

It's important to note that this paper does not delve into standard theories, as they are available in references [9] & [15]. The following section provides some well-known definitions for clarity.

***Definition* 1.1. [6]** The Nevanlinna characteristic function $T(r, f)$ for a meromorphic function f in the finite complex plane $\mathbb{C}$ is defined as:

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(re^{i\theta}\right)\right| d\theta + \int_0^r \frac{n(t,f)}{t} dt$$

where $n(t, f)$ represents the count of poles of the function f within the region $|z| \le t$ and

$$\begin{aligned}\log^+ x &= \log x \text{ when } x \ge 1\\ &= 0 \text{ when } 0 \le x \le 1.\end{aligned}$$

***Definition* 1.2. [6]** The order of a meromorphic function f is defined by

$$\rho = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

***Definition* 1.3. [4]** For a meromorphic function f having order zero, the quantity ρ^* is defined by

$$\rho^* = \limsup_{r\to\infty} \frac{T(r,f)}{\log r}.$$

***Definition* 1.4.** [13] A continuous function $L(r) > 0$ is said to be inereasing slowly if

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$$\lim_{r\to\infty}\frac{L(ar)}{L(r)} = 1 \ \forall\ a > 0.$$

***Definition* 1.5.** [14] For a meromorphic function f, its L-order ρ^L and L^*-order ρ^{L^*} are defined as

$$\rho^L = \limsup_{r\to\infty}\frac{\log\ T(r,f)}{\log\ (rL(r))}\ \text{and}\ \rho^{L^*} = \text{limsun}_{r\to\infty}\frac{\log\ T(r,f)}{\log\ [re^{L(r)}]}.$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

***Lemma* 2.1**. [12] For any complex number c_j with $|\arg\ c_j - \delta| \le \gamma \le \frac{\pi}{2}$ for some real γ and δ, the following inequality holds:

$$|c_j - c_{j-1}| \le ||c_j| - |c_{j-1}||\cos\ \gamma + (|c_j| + |a_{j-1}|)\sin\ \gamma.$$

***Lemma* 2.2.** [1] Let $g(z)$ be andytic in $|z| \le t$ with $g(0) = 0, g^J(0) = a$ and $|g(z)| \le K$ for $|z| = t$. Then for $|z| \le t$,

$$|g(z)| \le \frac{K|z|}{t^2}\cdot\frac{K|z| + t^2|a|}{K + |a||z|}.$$

3. Theorems

In this section, we present the main results of our research. These results significantly advance our understanding of the distribution of poles in meromorphic functions, particularly in the context of slowly changing functions. These offer valuable insights and solutions to address the challenges within this field, opening doors for further exploration and practical applications.

***Theorem* 3.1.** Let a meromorphic function $f(z)$ on $D \subseteq \mathbf{C}$ be of finite L-order $\rho^L(\ge 1)$ with $f(z) = \sum_{n-0}^{\infty} a_n z^n + \sum_{n=-1}^{-\infty} a_n z^n$ for $R_1 \le |z| \le R_2$. Also let for some real numbers γ and δ,

$$|\arg\ a_j - \delta| \le \gamma \le \frac{\pi}{2}, j = 0,1,2, \ldots$$

and

$$\rho^L|a_0| \ge R_2|a_1| \ge R_2^2|a_2| \ge \cdots.$$

Then poles of $f(z)$ reside in $D_1 \cup D_2$

where $D_1 = \left\{z \in D: R_2 < |z| \le \frac{BR_2}{|\rho_{a0} - R_2 a_1|}\right\}, D_2 = \{z \in D: |z| < R_1\}$

and $B = (\cos\ \gamma + \sin\ \gamma)\rho^L|a_0| + 2\sin\ \gamma\sum_{j=1}^{\infty}|a_j|R_2$.

***Proof*.** Clearly, $\lim_{n\to\infty} a_n R_2^n = 0$ and $\lim_{n\to\infty} a_n R_1^n = 0$.

Also, for $R_1 < |z| < R_2$, it follows that

$$|f(z)| \le |\textstyle\sum_{n=0}^{\infty}\ a_n z^n| + |\sum_{n=-1}^{\infty}\ a_n z^n|. \quad (1)$$

Now, for $|z| < R_2$, we get that

$(z - R_2)\sum_{n=0}^{\infty}\ a_n z^n$

$= -R_2 a_0 + (a_0 - R_2 a_1)z + \sum_{j=2}^{\infty}\ (a_{j-1} - R_2 a_j)z^j$

$= -R_2 a_0 + (a_0 - \rho^L a_0 + \rho^L a_0 - R_2 a_1)z + \sum_{j=2}^{\infty}\ (a_{j-1} - R_2 a_j)z^j$

$= -R_2 a_0 + a_0(1 - \rho^L)z + (\rho^L a_0 - R_2 a_1)z + \sum_{j=2}^{\infty}\ (a_{j-1} - R_2 a_j)z^j$

$= -R_2 a_0 + a_0(1 - \rho^L)z + G(z). \quad (2)$

Using *Lemma* 2.1, it follows for $|z| = R_2$ that

$$|G(z)| \le |\rho^L a_0 - R_2 a_1||z| + \sum_{j=2}^{\infty}|a_{j-1} - R_2 a_j||z|^j$$

$\le |\rho^L a_0 - R_2 a_1|R_2 + \sum_{j=2}^{\infty}\ |a_{j-1} - R_2 a_j|R_2$ $\le$ $\{|\rho^L|a_0| - R_2|a_1||\cos\gamma + (\rho^L|a_0| + R_2|a_1|)\sin\gamma\}R_2 + \sum_{j=2}^{\infty}\left\{\left||a_{j-1}| - R_2|a_j|\right|\cos\gamma + (|a_{j-1}| + R_2|a_j|)\sin\gamma\right\}R_2^j$

$= (\rho^L|a_0| - R_2|a_1|)R_2\cos\gamma + (\rho^L|a_0| + R_2|a_1|)R_2\sin\gamma + \sum_{j=2}^{\infty}\ (|a_{j-1}| - R_2|a_j|)R_2\cos\gamma$

$$+ \sum_{j=2}^{\infty}\ (|a_{j-1}| + R_2|a_j|)R_2^j\sin\gamma$$

$$= (\cos\gamma + \sin\gamma)\rho^L|a_0|R_2 + 2R_2\sin\gamma\sum_{j=1}^{\infty}|a_j|R_2^j$$

$$= R_2B \text{ where } B = (\cos\gamma + \sin\gamma)\rho^L|a_0| + 2\sin\gamma\sum_{j=1}^{\infty}|a_j|R_2^j.$$

Now, $G(z)$ being analytic in $|z| \le R_2$ with $G(0) = 0, G'(0) = (\rho' a_0 - R_2 a_1)$ and $|G(z)| \le R_2 B$ for $|z| = R_2$, we obtain by *Lemma* 2.2 that

$$|G(z)| \le \frac{BR_2|z|}{R_2^2}\cdot\frac{BR_2|z| + R_2^2 \mid \rho^L a_0 - R_2 a_1|}{BR_2 + |\rho^L a_0 - R_2 a_1||z|}$$

$$= \frac{B|z|\{B|z| + R_2|\rho^L a_0 - R_2 a_1|\}}{BR_2 + |\rho^L a_0 - R_2 a_1 \parallel z|}.$$

Hence for $|z| < R_2$, we get from (2) that

$$\left|(z - R_2)\sum_{n=0}^{\infty} a_n z^n\right|$$

$$\le |-R_2 a_0 + a_0(1-\rho^L)z| + \frac{B|z|\{B|z| + R_2|\rho^L a_0 - R_2 a_1|\}}{BR_2 + |\rho^L a_0 - R_2 a_1||z|}$$

$$\le \frac{(R_2|a_0| + |a_0|(\rho^L - 1)|z|)(BR_2 + |\rho^L a_0 - R_2 a_1||z|) + B|z|\{B|z| + R_2|\rho^L a_0 - R_2 a_1|\}}{BR_2 + |\rho^L a_0 - R_2 a_1||z|}$$

Therefore,

$$\left|\sum_{n=1}^{\infty} a_n z^n\right|$$

$$\le \frac{(R_2|a_0| + |a_0|(\rho^L - 1)|z|)(BR_2 + |\rho^L a_0 - R_2 a_1||z|) + B|z|\{B|z| + R_2|\rho^L a_0 - R_2 a_1|\}}{(R_2 - |z|)(BR_2 + |\rho^L a_0 - R_2 a_1||z|)}$$

$$\le \frac{(R_2|a_0| + |a_0|(\rho^L - 1)|z|)(BR_2 + |\rho^L a_0 - R_2 a_1||z|) + B|z|\{B|z| + R_2|\rho^L a_0 - R_2 a_1|\}}{(R_2 - |z|)(BR_2 - |\rho^L a_0 - R_2 a_1|z\,|)}. \quad (3)$$

Again, for $|z| > R_1$, we have

$$|(a - R_1)\textstyle\sum_{n=-1}^{-\infty} a_{a^2} a^n|$$

$$= \left|a_{-1} + (a_{-2} - R_1 a_{-1})\frac{1}{z} + (a_{-3} - R_1 a_{-2})\frac{1}{z^2} + \cdots\right|$$

$$\le |a_{-1}| + |(a_{-2} - R_1 a_{-1})|\left|\frac{1}{z}\right| + |(a_{-3} - R_1 a_{-2})|\left|\frac{1}{z}\right|^2 + \cdots$$

$$\le |a_{-1}| + (|a_{-2}| + R_1|a_{-1}|)\frac{1}{R_1} + (|a_{-3}| + R_1|a_{-2}|)\frac{1}{R_1} + \cdots$$

$$= 2\left(|a_{-1}| + \frac{|a_{-2}|}{R_1} + \frac{|a_{-3}|}{R_1^2} + \cdots\right)$$

$$= 2R_1\textstyle\sum_{n=-1}^{-\infty} |a_n|R_1^n = C.$$

Therefore, for $|z| > R_1$,

$$\left|\sum_{n=-1}^{-\infty} a_{n^2} z^n\right| \le \frac{C}{|z| - R_1} \quad (4)$$

Hence, by using (3) and (4), for $R_1 < |z| < R_2$ it follows

from (1) that

$$|f(z)| \le \frac{\left[\begin{matrix}(R_2|a_0| + |a_0|(\rho^L - 1)|z|)(BR_2 + |\rho^L a_0 - R_2 a_1||z|)\\ +B|z|\{B|z| + R_2|\rho^L a_0 - R_2 a_1|\}\end{matrix}\right]}{(R_2 - |z|)(BR_2 - |\rho^L a_0 - R_2 a_1||z|)} + \frac{C}{|z| - R_1}$$

$$\le \frac{\begin{matrix}(|z| - R_1)\{(R_2|a_0| + |a_0|(\rho^L - 1)|z|)(BR_2 + |\rho^L a_0 - R_2 a_1||z|)\\ +B|z|(B|z| + R_2|\rho^L a_0 - R_2 a_1|)\}\\ +C(R_2 - |z|)(BR_2 + |\rho^L a_0 - R_2 a_1 \parallel z|)\end{matrix}}{(R_2 - |z|)(BR_2 - |\rho^L a_0 - R_2 a_1||z|)(|z| - R_1)}.$$

Therefore,

$$\frac{1}{|f(z)|} > 0 \text{ if } (R_2 - |z|)(BR_2 - |\rho^L a_0 - R_2 a_1| \cdot |z|)(|z| - R_1) > 0.$$

Now, for $|z| > R_2$, it follows that $\frac{1}{|(z)}> 0$ if $(BR_2 - |\rho^L a_0 - R_2 a_1|.|z|) < 0$

$$\text{i.e., } \frac{1}{|f(z)\,|} > 0 \text{ if } |z| > \frac{BR_2}{\mid \rho^L a_0 - R_2 a_1|}.$$

Hence zeros of $\frac{1}{f(z)}$ reside in

$$R_2 < |z| \le \frac{BR_2}{|\rho^L a_0 - R_2 a_1|}.$$

As $f(z)$ is analytic in $R_1 \leq |z| \leq R_2$, poles of $f(z)$ lie in $D_1 = \left\{z \in D: R_2 < |z| \leq \frac{BR_3}{|\rho^L a_0 - R_2 a_1|}\right\}$.
Now, for $|z| < R_1 < R_2$, we se that

$$\frac{1}{|f(z)|} > 0 \text{ if } (BR_2 - |\rho^L a_0 - R_2 a_1||z|) < 0.$$

Consequently, poles of $f(z)$ reside in $D_2 = \{z \in D: |z| < R_1\}$.

Combining both the cases $D_1 \cup D_2$ is the region of poles of $f(z)$.

■

Remark 3.1. Taking $f(z)$ as a rational function and $L(r) = \log r, \rho^L = 0$. On the other hand, $\rho^* \geq 1$. Keeping all these in mind, the following theorem may state for meromorphic functions with $\rho^L = 0$.

***Theorem* 3.2.** Let a meromorphic function $f(z)$ on $D \subseteq \mathrm{C}$ be of finite order $\rho^*(\geq 1)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=-1}^{-\infty} a_n z^n$ for $R_1 \leq |z| \leq R_2$. Also let for some real numbers γ and δ,

$$|\arg\, a_j - \delta| \leq \gamma \leq \frac{\pi}{2}, j = 0,1,2, \ldots$$

and

$$\rho^*|a_0| \geq R_2|a_1| \geq R_2^2|a_2| \geq \cdots$$

Then poles of $f(z)$ reside in $D_1' \cup D_2'$
where $D_1' = \left\{z \in D: R_2 < |z| \leq \frac{B' R_2}{|\rho^* a_0 - R_2 a_1|}\right\}, D_2' = \{z \in D: |z| < R_1\}$
and $B' = (\cos\gamma + \sin\gamma)\rho^*|a_0| + 2\sin\gamma \sum_{j=1}^{\infty} |a_j| R_2^j$.

Theorem **3.2** can be proved as Theorem 3.1 and therefore its proof is excluded.

***Remark* 3.2**. The following example with related figure ensures the validity of Theorem 3.2.

***Example* 3.1**. Let

$$f(z) = \frac{1}{(z+i)(z+2i)(z+5)}.$$

Now for $2 < |z| < 5$, the Laurent's series expansion of $f(z)$ is

$$f(z) = \frac{23+15i}{29 \times 26 \times 5} - \frac{23+15i}{29 \times 26 \times 5^2} z + \frac{23+15i}{29 \times 26 \times 5^{3^2}} z^2 - \cdots + \frac{81-15i}{29 \times 26} \frac{1}{z} + \frac{115+75i}{29 \times 26} \frac{1}{z^2} + \cdots$$

Here,

$$a_0 = \frac{28+154}{23 \times 26 \times 5}, \qquad \mathrm{a}_1 = -\frac{23+154}{23 \times 26 \times 5^2}$$

and $\rho^* = 3$.

Taking $R_1 = 2.5,\ R_2 = 4,\ \gamma = \frac{\pi}{2}$ and $\delta = 0$, we see that all the conditions of Theorem 3.2 are satisfied.

Now,

$$B' = (\cos\gamma + \sin\gamma)\rho^*|a_0| + 2\sin\gamma \sum_{j=1}^{\infty} |a_j| R_2^j$$
$$= \frac{3}{29 \times 26 \times 5}|23+15i| + \frac{2}{29 \times 26 \times 5}|23+15i| \sum_{j=1}^{\infty} \left(\frac{4}{5}\right)^j \approx 0.08$$

and $|\rho^* a_0 - R_2 a_1| = \left|3\frac{23+15i}{29 \times 26 \times 5} + 4\frac{23+15i}{29 \times 26 \times 5^2}\right| \approx 0.028.$

Hence by Theorem 3.2, poles of $f(z)$ reside in

$$\{z \in \mathbb{C}: |z| < 2.5\} \cup \{z \in \mathbb{C}: 4 < |z| \leq 11.43\}.$$

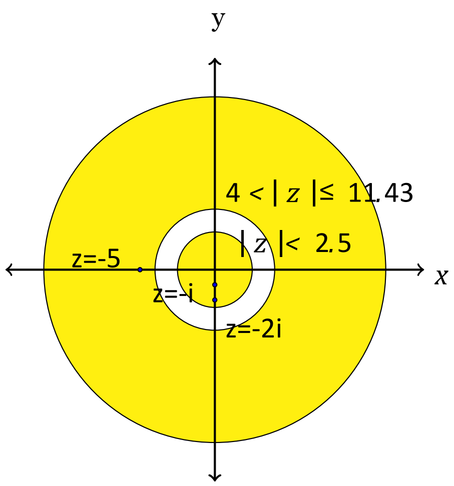

Figure 1: Distribution of the poles of $f(z) = \frac{23+15i}{29 \times 26 \times 5} - \frac{23+15i}{29 \times 26 \times 5^2} z + \frac{23+15i}{29 \times 26 \times 5^{3^2}} z^2 - \cdots + \frac{81-15i}{29 \times 26} \frac{1}{z} + \frac{115+75i}{29 \times 26} \frac{1}{z^2} + \cdots$

Continuing the discussion, the next theorem focuses solely on the real part of the coefficients of the analytic part of the Laurent series expansion of meromorphic functions.

***Theorem* 3.3.** Let a meromorphic function $f(z)$ on $D \subseteq \mathrm{C}$ be of finite L-onder $\rho^L(\geq 1)$ with

$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=-1}^{-\infty} a_n z^n$ for $R_1 \leq |z| \leq R_2$. If $a_j = \alpha_j + i\beta_j, j = 0,1,2,\dots$ and

$$0 < \rho^L \alpha_0 \geq R_2 \alpha_1 \geq R_2^2 \alpha_2 \geq \cdots$$

Then $D_3 \cup D_4$ is the region of poles of $f(z)$

where $D_3 = \left\{z \in D: R_2 < |z| \leq \frac{MR_2}{\rho^L \alpha_0 - R_2\alpha_1}\right\}, D_4 = \{z \in D: |z| < R_1\}$

and $M = \rho^L \alpha_0 + 2\sum_{j=1}^{\infty} |\beta_j| R_2^j$.

Proof. Clearly, $\lim_{j\to\infty} \alpha_j R_2{}^j = 0$, $\lim_{j\to\infty} \beta_j R_2^j = 0$ and $\lim_{j\to-\infty} a_j R_1^j = 0$.

Now, for $|z| < R_2$, it follows that

$$(z - R_2)\sum_{n=0}^{\infty} a_n z^n = -R_2 a_0 + (a_0 - R_2 a_1)z + \sum_{j=2}^{\infty} (a_{j-1} - R_2 a_j) z^j$$

$$= -R_2 a_0 + (\alpha_0 - R_2\alpha_1)z + i(\beta_0 - R_2\beta_1)z + \sum_{j=2}^{\infty} \{(\alpha_{j-1} - R_2\alpha_j) + i(\beta_{j-1} - R_2\beta_j)\} z^j$$

$$= -R_2 a_0 + (\alpha_0 - \rho^L\alpha_0 + \rho^L\alpha_0 - R_2\alpha_1)z + i(\beta_0 - R_2\beta_1) + \sum_{j=2}^{\infty} \{(\alpha_{j-1} - R_2\alpha_j) + i(\beta_{j-1} - R_2\beta_j)\} z^j$$

$$= -R_2 a_0 + (1 - \rho^L)\alpha_0 z + (\rho^L\alpha_0 - R_2\alpha_1)z + i(\beta_0 - R_2\beta_1)z + \sum_{j=2}^{\infty} \left\{(\alpha_{j-1} - R_{2\alpha_j}) + i(\beta_{j-1} - R_2\beta_j)\right\} z^j$$

$$= -R_2\alpha_0 + (1 - \rho^L)\alpha_0 z + H(z). \quad (1)$$

For $|z| = R_2$, we have

$$|H(z)| \leq |\rho^L \alpha_0 - R_2 \alpha_1||z| + |\beta_0 - R_2 \beta_1||z| + \sum_{j=2}^{\infty} |\alpha_{j-1} - R_2\alpha_j||z|^j + \sum_{j=2}^{\infty} |\beta_{j-1} - R_2\beta_j||z|^j$$

$$\leq (\rho^L\alpha_0 - R_{2\alpha_1})R_2 + (|\beta_0| + R_2|\beta_1|)R_2 + \sum_{j=2}^{\infty} (\alpha_{j-1} - R_2\alpha_j)R_2 + \sum_{j=2}^{\infty} (|\beta_{j-1}| + R_2|\beta_j|)R_2$$

$$= \rho^L\alpha_0 R_2 + R_2|\beta_0| + 2R_2\sum_{j=1}^{\infty} |\beta_j| R_2^j$$

$= MR_2$ where $M = \rho^L\alpha_{\alpha_0} + |\beta_0| + 2\sum_{j=1}^{\infty} |\beta_j| R_2^j$.

As $H(z)$ is analytic in $|z| \leq R_2$ with $H(0) = 0, H'(0) = (\rho^L \alpha_0 - R_2\alpha_1)$ and $|H(z)| \leq R_2 M$ for $|z| = R_2$, we get by Lemma 2.2 that

$$|H(z)| \leq \frac{MR_2|z|}{R_2^2} \cdot \frac{MR_2|z| + R_2^2 |\rho^L\alpha_0 - R_2\alpha_1|}{MR_2 + |\rho^L\alpha_0 - R_2\alpha_1||z|} = \frac{M|z|\{M|z| + R_2(\rho^L\alpha_0 - R_2\alpha_1)\}}{MR_2 + (\rho^L\alpha_0 - R_2\alpha_1)|z|}$$

Hence for $|z| < R_2$, it follows from (1) thut

$$|(z - R_2)\sum_{n=0}^{\infty} a_n z^n|$$

$$\leq |-R_2 a_0 + (1 - \rho^L)\alpha_0 + \frac{M|z|\{M|z| + R_2(\rho^L\alpha_0 - R_2\alpha_1)\}}{MR_2 + (\rho^L\alpha_0 - R_2\alpha_1)|z|}$$

$$\leq R_2|a_0| + (\rho^L - 1)\alpha_0|z| + \frac{M|z|\{M|z| + R_2(\rho^L\alpha_0 - R_2\alpha_1)\}}{MR_2 + (\rho^L\alpha_0 - R_2\alpha_1)|z|}$$

$$= \frac{\{R_2|a_0| + (\rho^L - 1)\alpha_0|z|\}\{MR_2 + (\rho^L\alpha_0 - R_2\alpha_1)|z|\} + M|z|\{M|z| + R_2(\rho^L\alpha_0 - R_2\alpha_1)\}}{MR_2 + (\rho^L\alpha_0 - R_2\alpha_1)|z|}$$

Therefore,

$$\left|\sum_{n=0}^{\infty} a_n z^n\right| \leq \frac{\{R_2|a_0| + (\rho^L - 1)\alpha_0|z|\}|MR_2 + (\rho^L\alpha_0 - R_2\alpha_1)|z|| + M|z|\{M|z| + R_2(\rho^L\alpha_0 - R_2\alpha_1)\}}{(R_2 - |z|)\{MR_2 + (\rho^L\alpha_0 - R_2\alpha_1)|z|\}}$$

$$\leq \frac{\{R_2|a_0| + (\rho^L - 1)\alpha_0|z|\}|MR_2 + (\rho^L\alpha_0 - R_2\alpha_1)|z|| + M|z|\{M|z| + R_2(\rho^L\alpha_0 - R_2\alpha_1)\}}{(R_2 - |z|)\{MR_2 - (\rho^L\alpha_0 - R_2\alpha_1)|z|\}} \quad (2)$$

Now, for $|z| > R_1$, we get from (4) of Theorem 3.1 that

$$\left|\sum_{n=-1}^{-\infty} a_n z^n\right| \leq \frac{C}{|z| - R_1} \text{ where } C = 2R_1\sum_{n=-1}^{-\infty} |a_n| R_1^n. \quad (3)$$

Hence, we obtain for $R_1 < |z| < R_2$ that

$$|f(z)| \leq \left|\sum_{n=0}^{\infty} a_n z^n\right| + \left|\sum_{n=-1}^{-\infty} a_n z^n\right|$$

$$\leq \frac{\{R_2|a_0| + (\rho^L - 1)\alpha_0|z|\}\{MR_2 + (\rho^L\alpha_0 - R_2\alpha_1)|z|\} + M|z|\{M|z| + R_2(\rho^L\alpha_0 - R_2\alpha_1)\}}{(R_2 - |z|)\{MR_2 - (\rho^L\alpha_0 - R_2\alpha_1)|z|\}}$$

$$+ \frac{C}{|z| - R_1}$$

$$\leq \frac{(|z| - R_1)[\{R_2|a_0| + (\rho^L - 1)\alpha_0|z|\}\{MR_2 + (\rho^L\alpha_0 - R_2\alpha_1)|z|\} + M|z|\{M|z| + R_2(\rho^L\alpha_0 - R_2\alpha_1)\}] \cdot C(R_2 - |z|)(MR_2 - (\rho^L\alpha_0 - R_2\alpha_1)|z|)}{(|z| - R_1)(R_2 - |z|)\{MR_2 - (\rho^L\alpha_0 - R_2\alpha_1)|z|\}} \tag{4}$$

From (4), we see that

$$\frac{1}{|f(z)|} > 0 \text{ if } (|z| - R_1)(R_2 - |z|)\{MR_2 - (\rho^L\alpha_0 - R_2\alpha_1)|z|\} > 0.$$

Now, for $|z| > R_2$,

$$\frac{1}{|f(z)|} > 0 \text{ if } MR_2 - (\rho^L\alpha_0 - R_2\alpha_1)|z| < 0$$

$$\text{i.e., } \frac{1}{|f(z)|} > 0 \text{ if } |z| > \frac{MR_2}{\rho^L\alpha_0 - R_1\alpha_1}.$$

Hence zeros of $\frac{1}{f(z)}$ reside in $R_2 < |z| \leq \frac{MR_2}{\rho^L\alpha_0 - R_1\alpha_1}$ and consequently poles of $f(z)$ in $R_2 < |z| \leq \frac{MR_2}{\rho^L\alpha_0 - R_1\alpha_1}$.

Also, for $|z| < R_1 < R_2$,

$$\frac{1}{|f(z)|} > 0 \text{ if } \{MR_2 - (\rho^L\alpha_0 - R_2\alpha_1)|z|\} < 0.$$

Therefore, poles of $f(z)$ reside in $|z| < R_1$. Thus the theorem is established.

■

The forthcoming theorem shares similarities with the previous one and is applicable to meromorphic functions with $\rho^L = 0$.

***Theorem* 3.4.** Let a meromorphic function $f(z)$ on $D \subseteq \mathbb{C}$ be of finite order $\rho^*(\geq 1)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=-1}^{-\infty} a_n z^n$ for $R_1 \leq |z| \leq R_2$. If $a_j = \alpha_j + i\beta_j, j = 0, 1, 2, \ldots$ and

$$0 < \rho^*\alpha_0 \geq R_2\alpha_1 \geq R_2^2\alpha_2 \geq \cdots$$

Then poles of $f(z)$ reside in $D_3' \cup D_4'$

where $D_3' = \left\{z \in D: R_2 < |z| \leq \frac{M'R_2}{\rho^*\alpha_0 - R_2\alpha_1}\right\}, D_4' = \{z \in D: |z| < R_1\}$ and $M' = \rho^*\alpha_0 + 2\sum_{j=1}^{\infty} |\beta_j| R_2^j$.

The proof is similar to Theorem 3.3.

***Remark* 3.3**. The following example with related figure justifies the validity of Theorem 3.4.

***Example* 3.2**. Let

$$f(z) = \frac{1}{(z-1)(z-2)(3-z)}.$$

Now, for $2 < |z| < 3$, the Laurent's series expansion of $f(z)$ is

$$f(z) = \frac{1}{6} + \frac{1}{18}z + \frac{1}{54}z^2 + \cdots + \frac{1}{2z} + \frac{3}{2z^2} + \cdots$$

Here, $R_1 = 2.01, R_2 = 2.99,\ \alpha_0 = \frac{1}{6},\ \alpha_1 = \frac{1}{18}$ and $\rho^* = 3$.

Now,

$$M' = \rho^*\alpha_0 + 2\sum_{j=1}^{\infty} |\beta_j| R_2^j$$

$$= \frac{1}{2} \quad \text{and} \quad \frac{M'R_2}{\rho^*\alpha_0 - R_2\alpha_1} \approx 4.48$$

Hence by Theorem 3.4, the region for poles of $f(z)$ is

$$\{z \in \mathbb{C}: |z| < 2.01\} \cup \{z \in \mathbb{C}: 2.99 < |z| \leq 4.48\}.$$

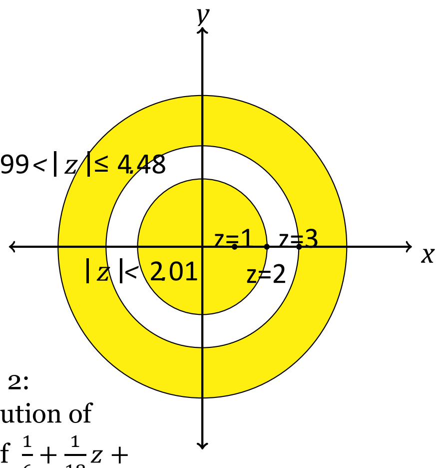

Figure 2: Distribution of poles of $\frac{1}{6} + \frac{1}{18}z + \frac{1}{54}z^2 + \cdots + \frac{1}{2z} + \frac{3}{2z^2} + \cdots$

***Remark* 3.4**. Theorem 3.1 and Theorem 3.3 are also valid for meromorphic functions with $\rho^{L^*} \geq 1$.

Future prospect.

In the line of the works as carried out in the paper one may think of proving the results in case of meromorphic functions having infinite L^*-order.

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