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Dynamics of a density dependent stage structure prey-predator model under fuzzy and stochastic impreciseness

Prasenjit Mahato[1], Subhashis Das[1], Sanat Kumar Mahato[1,*], Partha Karmakar[2], Pintu Pal[3]

[1] Department of Mathematics, Sidho-Kanho-Birsha University, Purulia, West Bengal-723104, INDIA
[2] Deputy Secretary, West Bengal Board of Primary Education, Salt Lake City, Kolkata, West Bengal-700091, INDIA
[3] Department of Computer Application, Asansol Engineering College, Asansol, West Bengal-713305, INDIA

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Abstract

The interdisciplinary natural problems are commonly solved by using mathematical models in prey predator interplay. In this work we propose fuzzy stochastic prey predator model with stage structure. The model consist prey, immature, mature population. Firstly, we have constructed the crisp model with considering some assumptions and notations. It is transformed to the fuzzy model. All the controlled biological parameters are taken in imprecise nature and considered as triangular fuzzy numbers. We adopt the graded mean integration formula for the defuzzification of the fuzzy prey predator model to get the solution easily. Then we form it in stochastic model with the help of Markov chain process and also investigate the persistence in mean and extinction of the system. The positivity, boundedness, equilibria, global stability of the interior equilibrium point and Hopf bifurcation analysis are discussed for the proposed model. We use MATLAB package for numerical experiment. We represent the phase space trajectories for different values of parameters in crisp, fuzzy environment. Also, we have shown graphically the prey predator limit cycle in stochastic. The evolution in time of the predator and of the prey with different noise is presented graphically. The bifurcation scenarios are presented with the help of MATCONT packages. Lastly, we describe the sensitivities of the controlled parameters.

Keywords: Density dependent effect, Triangular Fuzzy Number, Graded Mean Integration Value, Stochastic model, Hopf bifurcation

* Corresponding author
Email: pmmath1994@skbu.ac.in, dassubhashis409@skbu.ac.in, sanatkr_mahato.math@skbu.ac.in, partha_math72@yahoo.co.in, pintupalaec@gmail.com

1. Introduction

Prey predator dynamic is the important chapter in the mathematical ecology, basically for our understanding of interacting populations in the environment. There is huge diversity of the plants, insects and animals in their life histories. In recent years, researchers focus on evaluating population and interaction among them. At first Lotka and Volterra introduced the predator prey model [17, 20, 24, 32, 40] and they have interested many researchers in ecology. Species carry on their life time through several life stages such as immature and mature stages in the natural world. So, predator-prey model with stage structure [1-3, 5, 10, 12, 15, 16, 25, 26, 33, 34, 48, 51-53, 55] is more adequate. Bai et al. [1] investigated the stability and Hopf bifurcation for a predator-prey model with additional food for predator. The dynamics predator prey model with stage structure on both species anti predator behavior was proposed by Mortoja et al. [31]. In their work they discussed the periodic oscillation of adult predator population and described the bifurcation diagram for the parameters. A prey predator model with stage structured for predator population and harvesting for mature predator population has been investigated by Chakraborty et al. [4]. In this article we analyze the dynamic model of prey predator system. When the prey cannot fulfill the demand for food of predator, the predator looks for alternative source of food [19, 22, 46, 47]. In proposed model the parameters and initial conditions are considered as crisp or fixed. The experimental variables and parameters may be varied in real world system for our error look out. Fuzzy sets and fuzzy logic is one of the important tool to consider it properly. Chang and Zadeh [6, 54] first introduced the idea of fuzzy derivative.

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Kalev et al. [21] first discussed the concept of differential equations. With the help of generalized Hukuhara derivative [36, 39, 41, 50] the existence and the uniqueness of solution for fuzzy initial value problems were studied by Villamizar-Roa et al. [49]. Guo et al. [18] proposed the oscillation of delay differential inclusions and fuzzy biodynamics model. Das et al. and Mahato et al. [8] investigated the prey predator model in controlling disease with prey refuge under the fuzzy environment. They discussed the global stability analysis and bifurcation analysis of the model. They also explained the entire figure in crisp environment and fuzzy environment. Many researchers studied the model for species under fuzzy environment [37, 38] and used fuzzy parameter [35] to discuss their work.

The parameters involved in the mathematical models on prey predator model discussed are crisp in nature and also it is transformed to fuzzy model. We use graded mean integration technique [28] to transform the fuzzy model into the defuzzified model. All the biological parameters are taken as triangular fuzzy number. As real life is full of randomness and stochasticity, we have transformed crisp model as stochastic model. Many researchers have investigated the stochastic models with white noise perturbations, refers to [3, 13, 14, 23, 27]. Das et.al [7] described a prey predator model in case of disease transformation via pest under uncertainty. They explained crisp model, fuzzy model, defuzzified model and stochastic model with white noise in their research work. Das et al. [11] proposed the stochastic prey predator model with additional food. In his work they showed the perturbation with white noise for prey's growth rate and predator's death rate. A stochastic prey predator model with time dependent delay was investigated by Dai et al. [9]. They discussed the positivity, global solution and stochastically boundedness of the model. They also investigated the persistence in mean and extinction of the system. Schwartz et al. [45] proposed a population dynamical model. They included the influence of delay on the rate of noise induced switching between stable states and noise induced extinction in research work. A theory on the ecological web testing the effect of environmental noise on population was studied by Ripa et al. [43]. The substance of randomly changeable in environment can fluctuate birth rates, death rates, carrying capacity and all other parameter involved in model was described by May et al. [30].

We make up the remaining work in several sections and subsections. We have some basic assumptions and notation for using our work. Some useful preliminary concepts are discussed. We divide model formulation section by the subsection of crisp model, fuzzy model, defuzzified model and stochastic model. Positivity of the model, boundedness of the system, equilibria, and stability analysis are investigated in theoretical study section. Also, Hopf bifurcation analysis of the proposed model is discussed. Main findings of analytical results are verified and described in numerical results section. Some numerical examples are in the section of numerical results. We describe the sensitivities of the different parameters and draw the phase portrait of the system in both crisp and fuzzy environment. The conclusion portion is discussed in last section.

2. Assumptions and Notation

We use some basic assumptions and notation in our work.

Assumptions:

i. Prey and Predator both are present in the population.
ii. Predator population is divided into two parts, immature predator and mature predator.
iii. Only prey population is reproducing according to logistic law.
iv. Only mature predator has maximum growth rate due to alternative source food.
v. There is natural death for both predators.
vi. There is conversion factor from prey population to mature predator population.
vii. Immature predators reproduce from mature predators.
viii. Immature predators translate to mature predators.
ix. There is density dependent effect to the focal prey population
x. There is no natural death rate for prey population.

Notation:

$\tilde{A}$ Fuzzy set

$\mu_{\tilde{A}}(x)$ Membership function for the fuzzy set $\tilde{A}$

$G(\tilde{A})$ Graded mean integral value of $\tilde{A}$

$L(x)$ Left shape function of $\tilde{A}$

$R(x)$ Right Shape function of $\tilde{A}$
w Degree of optimism
$x(t)$ Prey population
$y(t)$ Immature predator population
$z(t)$ Mature predator population
γ Intrinsic growth rate of prey population
k Carrying capacity
α Predation rate for prey population
s Reproduction rate of mature predator
β Translation rate from immature predators to mature predators
ε_1 Natural death rate of immature predators
m Conversion factor rate from prey to mature predator
ε_2 Natural death rate of mature predator
μ Maximum growth rate of mature predators

3. Preliminaries

Fuzzy Set: Let the collection of the object be X and x be the elements of X. Then a fuzzy set $\tilde{A}$ in X is a set of order pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)): x \epsilon X\}$, where $\mu_{\tilde{A}}(x)$ is called the membership function of x in $\tilde{A}$ which maps X to [0,1].

Triangular Fuzzy Number (TFN): A triangular fuzzy number is specified by the ordered triplet (a, b, c) (**Fig. 1**) and defined by its continuous membership function $\mu_{\tilde{A}}(x)$: X→ [0,1] as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & if\ x = b \\ \frac{c-x}{c-b} & if\ b \le x \le c \\ 0 & otherwise \end{cases}$$

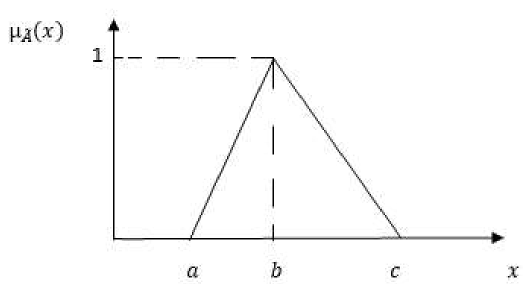

Fig. 1: Triangular Fuzzy Number

Operations of Triangular Fuzzy Number: The following are the four operations that can be performed on triangular fuzzy numbers: Let $\tilde{A}$ = (a_1, a_2, a_3) and $\tilde{B}$= (b_1, b_2, b_3) then,

i. Addition: $\tilde{A} + \tilde{B} = (a_1+b_1, a_2+b_2, a_3+b_3)$,
ii. Subtraction: $\tilde{A} - \tilde{B} = (a_1- b_3, a_2- b_2, a_3- b_1)$,
iii. Multiplication: $\tilde{A} \times \tilde{B}$ = (min $(a_1b_1, a_1b_3, a_3b_1, a_3b_3)$, a_2b_2, max $(a_1b_1, a_1b_3, a_3b_1, a_3b_3)$),
iv. Division: $\tilde{A}/\tilde{B}$ = (min $(a_1/b_1, a_1/b_3, a_3/b_1, a_3/b_3)$, a_2/b_2, max$(a_1/b_1, a_1/b_3, a_3/b_1, a_3/b_3)$), $b_i \neq 0$, i=1, 2, 3.

Graded Mean Integration Representation of Fuzzy Number: The graded mean integral value of the fuzzy number Ã is defined as

$$G(\tilde{A}) = \frac{\int_0^1 x\{(1-w)L^{-1}(x)+wR^{-1}(x)\}dx}{\int_0^1 x dx}$$
$$=2\int_0^1 x\{(1-w)L^{-1}(x) + wR^{-1}(x)\}dx$$

where, $L(x)$ and $R(x)$ are the left and right shape functions of $\tilde{A}$.

GMIV Formula for Triangular Fuzzy Number (TFN): Here we discuss the GMIV formula [28]. For the TFN Ã, the left and right shape functions are respectively, L(x) = $\frac{x-a_1}{a_2-a_1}$ and R(x) =$\frac{a_3-x}{a_3-a_2}$. Therefore, $L^{-1}(x) = a_1 + (a_2-a_1)$ x and R^{-1}(x) = a_3- $(a_3-a_2)\,x$.

Now, GMIV of Ã

$$= 2\int_0^1 x\{(1-w)L^{-1}(x) + wR^{-1}(x)\}dx$$
$$= 2\int_0^1 x\{(1-w)[a_1 + (a_2 - a_1)\,x] + w[a_3 - (a_3 - a_2)x]\}dx$$
$$= 2[(1\text{-}w)\,\{\frac{a_2}{2} + \frac{(a_2-a_1)x}{3}\} + w\{\frac{a_3}{2} - \frac{(a_3-a_2)}{3}\}]$$
$$= \frac{1}{3}[(1-w)a_1 + 2a_2 + wa_3]$$

Thus, $G(\tilde{A}) = \frac{1}{3}[(1-w)a_1 + 2a_2 + wa_3]$.
If $\tilde{A} = (a_1, a_2, a_3)$ is a triangular fuzzy number, then, $G(\tilde{A})$ reduces to the real number a.

4. Model Formulation

Case-I: Crisp Model

In this paper of prey predator system, we fit three different species; they are the prey, the immature predators and the mature predators. We denote $x(t)$ as the population size of prey, $y(t)$ as the population size of immature predators and $z(t)$ as the population size of matured predators at any time t. Therefore, in order to describe the dynamics of this model mathematically, the following hypotheses are adopted.

a) Let γ be the intrinsic growth rate of the prey population which grows logistically and let k be the environmental carrying capacity. According to mass action law, the matured predator's population attacks the prey population with the predation rate α (>0).
b) Let s be the matured predators population reproduction rate and β (>0) be the translation rate from the immature predator's population to matured predators' population. Also, let the natural death rate for the immature predators be ε_1 (>0).
c) Let the conversion factor from prey population to the matured predator's population is m and again ε_2(>0) be the natural death rate of the matured predators. Also, let us assume that μ is the maximum growth rate due to alternative source of food. The density dependent effect to the focal prey population is represented by the term $(1 - x/k)$. If the focal prey population x increases, the predator uses less amount of alternative source. The consumption of alternative source tends to zero when x approaches k.

Under the above assumptions, the prey predator model can be represented by the following differential equations

$$\frac{dx}{dt} = \gamma x\left(1 - \frac{x}{k}\right) - \alpha xz$$
$$\frac{dy}{dt} = sz - \beta y - \varepsilon_1 y$$
$$\frac{dz}{dt} = mxz + \beta y - \varepsilon_2 z + \mu z(1 - x/k) \qquad (1)$$

where, the initial conditions are $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$. **Table 1** presents units or dimension of the variables or parameters.

Table 1 Units or dimensions of the variables and parameters

Variables/ parameters	Units/ dimensions	Variables/ parameters	Units/ dimension
$x(t)$	Mass	s	Mass per unit time
$y(t)$	Mass	β	Mass per unit time
$z(t)$	Mass	ε_1	Mass per unit time
γ	Mass per unit time	m	Dimensionless
k	Mass	ε_2	Mass per unit time
α	Mass per unit time	μ	Mass per unit time

Case-II: Fuzzy Model

We assume that biological parameters are fuzzy numbers represented by TFNs. Assuming the biological parameters are fuzzy in nature, the crisp model (1) becomes

$$\frac{\widetilde{dx}}{dt} = \tilde{\gamma}x(1 - x/\tilde{k}) \ominus \tilde{\alpha}xz$$
$$\frac{\widetilde{dy}}{dt} = \tilde{s}z \ominus \tilde{\beta}y \ominus \tilde{\varepsilon_1}\, y$$
$$\frac{\widetilde{dz}}{dt} = \tilde{m}xz \oplus \tilde{\beta}y \ominus \tilde{\varepsilon_2}\, z \oplus \tilde{\mu}z(1 - x/\tilde{k}) \qquad (2)$$

where, $\tilde{\gamma}, \tilde{k}, \tilde{\alpha}, \tilde{s}, \tilde{\beta}, \tilde{\varepsilon_1}, \tilde{m}, \tilde{\varepsilon_2}, \tilde{\mu}$ are all triangular fuzzy numbers.

Case- III: Defuzzified Model

We defuzzified the fuzzy model by using graded mean defuzzification method. Then the above model (2) is represented as following form

$$G(\frac{\widetilde{dx}}{dt}) = G(\tilde{\gamma})x(1 - x/G(\tilde{k})) \ominus G(\tilde{\alpha})xz$$

$$G(\frac{\widetilde{dy}}{dt}) = G(\tilde{s})z \ominus G(\tilde{\beta})y \ominus G(\tilde{\varepsilon_1})\, y$$

$$G\left(\frac{\widetilde{dz}}{dt}\right) = \tilde{m}xz \oplus \tilde{\beta}y \ominus \tilde{\varepsilon_2}\, z \oplus \tilde{\mu}z(1 - x/\tilde{k}) \qquad (3)$$

where, $G(\ )$'s are the defuzzified values of the corresponding fuzzy numbers.

Case- IV: The Stochastic Model

In the study of the biological process the stochastic models are the more realistic models. The growth rate and natural death rate of prey-predator are randomly changed by the natural disaster and human disturbances. Hence the white noise is considered to be the most important tool to explain the randomly fluctuating phenomena. Here the crisp model (1) is transformed into the stochastic prey predator model. In this portion we have described that white noise get into the whole population dynamics. We have considered the white noise perturbation to the intrinsic growth rate of prey population γ, natural death rate of immature predator ε_1 and natural death rate of mature predator ε_2. That is $\gamma \rightarrow \gamma + \theta_1\ \dot{H}_1(t)$,

$-\varepsilon_1 \to -\varepsilon_1 + \Theta_2 \; \dot{H}_2(t)$, $-\varepsilon_2 \to -\varepsilon_2 + \Theta_3 \; \dot{H}_3(t)$, where the strength of noise are represented by Θ_1,Θ_2, Θ_3 and the mutually independent Brownian motions or standard Weiner process are represented by $H_1(t), H_2(t), H_3(t)$. A continuous time Markov Chain $f(t)$ can explain it and it takes a finite value in the state space $M = \{1, 2, \ldots 3\}$.Then the model (1) reduces in the following forms:

$$dx(t) = x(t)\left[\gamma(f(t))\left(1-\frac{x(t)}{k(f(t))}\right) - \alpha(ft)ztdt+\theta1(f(t))x(t)dH1(t)\right.$$

$$dy(t) = [\,s(f(t))z(t) - \beta(f(t))y(t) - \varepsilon1ftytdt+\theta2(f(t))y(t)dH2(t)$$

$$dz(t) = z(t)\Big\{\Big[m(f(t))x(t) - \varepsilon_2(f(t)) + \mu ft1-xtkgt+\beta ftytdt+ \theta3(f(t))z(t)dH_3(t) \qquad (4)$$

We choose the generator $F = (w_{ij})_{n\times n}$ that can be defined as

$$P\{f(t+\Delta t) = j | f(t) = i\} = \begin{cases} w_{ij}\Delta t + o(\Delta t), & if\ i \neq j \\ 1 + w_{ij}\Delta t + o(\Delta t), & if\ i = j \end{cases}$$

where, Δt is positive and $w_{ij}(\geq 0)$ represents the transition rate which runs from i to j and the condition applied $\sum_{i=1}^{n} w_{ij} = 0$. Considering Markov Chain $f(t)$ is irreducible. Above system (4) becomes

$$dx(t) = x(t)\left[\gamma(j)\left(1-\frac{x(t)}{k(j)}\right) - \alpha(j)z(t)\right]dt + \Theta_1(j)x(t)dH_1(t)$$

$$dy(t) = [\,s(j)z(t) - \beta(j)y(t) - \varepsilon_1(j)y(t)]dt + \Theta_2(j)y(t)dH_2(t)$$

$$dz(t) = z(t)\Big\{\Big[m(j)x(t) - \varepsilon_2(j) + \mu(j)\Big(1 - xtkj+\beta jytdt+\theta3(j)z(t)dH3(t)$$ while $j\in x$.

This Markov Chain has unique stationary distribution $\Omega = \{\Omega_1, \Omega_2 \ldots . \Omega_n\}$. It has the solution of the system of equation $\Omega F = 0$. It satisfy the condition $\sum_{i=1}^{n} \Omega_i = 1$ and $\Omega_i > 0$. We choose

$$\chi_1(f(t)) = \gamma(f(t)) - \frac{\Theta_1{}^2(f(t))}{2},$$

$$\chi_2(f(t)) = -\varepsilon_1(f(t)) - \frac{\Theta_2{}^2(f(t))}{2},$$

$$\chi_3(f(t)) = -\varepsilon_2(f(t)) - \frac{\Theta_3{}^2(f(t))}{2}$$

for simplification.

We prove the theorem with the help of the following definition.

Persistence and Extinction:

(1) If the condition $\liminf_{t\to\infty} \frac{\int_0^t g(s)ds}{t} > 0$ a.s. is satisfied then $g(t)$ is called strongly persistence in mean.

(2) If the condition $\lim_{t\to\infty} g(t) = 0$ a.s. is satisfied then $g(t)$ is called extinct.

Theorem 1

i) If the condition $\sum_{i=1}^{n} \Omega_i(\omega_1(i) + \beta(i)) < 0$ is satisfied then the prey population in the environment is extinct.

ii) If the condition $\sum_{i=1}^{n} \Omega_i(\omega_2(i) + s(i)k(i)) < 0$ is satisfied then the immature predator population in environment is extinct.

iii) If the condition $\sum_{i=1}^{n} \Omega_i(\omega_3(i) + m(i)\alpha(i)k(i)) < 0$ is satisfied then the mature predator population in environment is extinct.

Proof: i) We have applied Itô's formula in first equation of the model system (4), then we have

$$d\, In\, x(t) \leq (\omega_1(f(t)) + \beta(f(t))))dt + \theta_1(f(t))dH_1(t);$$

These gives

$$\frac{In\, x(t) - In\, x(0)}{t} = \frac{\int_0^t [\omega_1(f(t)) + \beta(f(t)))]dt}{t} + \frac{\int_0^t \theta_1(f(t))dH_1(t)dt}{t}.$$

According to the ergodic theory of Markov chain and strong law of large number, we get

$$\lim_{t\to\infty} sup\, \frac{In\, x(t)}{t} \leq \sum_{i=1}^{n} \Omega_i(\omega_1(i) + \beta(i))$$

If the condition$\sum_{i=1}^{n} \Omega_i(\omega_1(i) + \beta(i)) < 0$ is satisfied then we have $\lim_{t\to\infty} x(t) = 0$ a.s. that means prey population $x(t)$ in the environment is extinct.

ii) In analogous way, we have from the second equation of the model system (4)

$d\, In\, y(t) \leq (\omega_2(f(t)) + s(f(t))k(f(t)))dt + \theta_2(f(t))dH_2(t);$

This implies

$$\frac{In\, y(t) - In\, y(0)}{t} \leq \frac{\int_0^t (\omega_2(f(t)) + s(f(t))k(f(t)))dt)}{t} + \frac{\int_0^t \theta_2(f(t))dH_2(t)}{t};$$

Therefore $\lim_{t\to\infty} sup \frac{In\, y(t)}{t} \leq \sum_{i=1}^{n} \Omega_i\, (\omega_2(i) + s(i)k(i)$.

If the condition $\sum_{i=1}^{n} \Omega_i\, (\omega_2(i) + s(i)k(i) < 0$ is satisfied then we have $\lim_{t\to\infty} y(t) = 0$ a.s. Hence immature predator population in environment is extinct.

iii) Again, we have got from the third equation of the model system (4),

$d\, In\, z(t) \leq ((\omega_3(f(t)) + m(f(t))\alpha(f(t))k(f(t)))\, dt + \theta_3(f(t))dH_3(t);$

This implies

$$\frac{In\, z(t) - In\, z(0)}{t} \leq \frac{\int_0^t (\omega_3(f(t)) + m(f(t))\alpha(f(t))k(f(t)))dt}{t} + \frac{\int_0^t \theta_3(f(t))dH_3(t)}{t}$$

This gives

$$\lim_{t\to\infty} sup \frac{In\, z(t)}{t} \leq \sum_{i=1}^{n} \Omega_i\, (\omega_3(i) + m(i)\alpha(i)k(i))$$

If the condition $\sum_{i=1}^{n} \Omega_i\, (\omega_3(i) + m(i)\alpha(i)k(i)) < 0$ is satisfied then we have $\lim_{t\to\infty} z(t) = 0$ a.s. Hence mature predator population in the environment is extinct.

5. Dynamical study of the model:

In this section we describe the positivity, boundedness of the system, equilibria and the stability analysis of the model.

Positivity of the model

In this section, we desire to show that the solutions of the system of differential equations (1) are always nonnegative.

The vector formation of the system (1) is

$$\dot{V} = \varphi(V(t)) \tag{5}$$

where, V(t) $=(v_1, v_2, v_3)^T = (x(t), y(t), z(t))^T$, V (0) $=(x(0), y(0), z(0))^T \in R_+^3$ and

$$\varphi(V(t)) = \begin{pmatrix} \varphi_1(V(t)) \\ \varphi_2(V(t) \\ \varphi_3(V(t) \end{pmatrix}$$

$$= \begin{pmatrix} \gamma x\left(1 - \frac{x}{k}\right) - \alpha xz \\ sz - \beta y - \varepsilon_1 y \\ mxz + \beta y - \varepsilon_2 z + \mu z(1 - x/k) \end{pmatrix}$$

It can be shown that $[\varphi_i(V(t))]_{v_i=0} \geq 0$, (for i=1, 2, 3). For solving uniqueness of the system of differential equations we use Nagumo's theorem. So any solution of (5) with initial point $V(0) = V_0 \in R_+^3$ can be written as V(t) = V(t; V_0), where V(t)$\in R_+^3$ for all t> 0. i.e., the solutions of the system (1) remain non negative throughout the region R_+^3.

Boundedness of the system:

Theorem 2: All the solutions (x(t), y(t), z(t)) of the system (1) with positive initial condition are uniformly bounded.

Proof: Let us define the positive definite function W as

$$W = x + y - z$$

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{dy}{dt} - \frac{dz}{dt}$$

$$= \gamma x\left(1 - \frac{x}{k}\right) - (\alpha + m)xz - \varepsilon_1 y + (s + \varepsilon_2)z - \mu z(1 - \frac{x}{k})$$

$$\leq \gamma x\left(1 - \frac{x}{k}\right) - (\alpha + m)xz - K_1 W - \mu z(1 - \frac{x}{k}),$$

where $K_1 = min(\varepsilon_1, (s + \varepsilon_2))$, arbitrary constant.

Hence, $\frac{dW}{dt} + K_1 W \leq \gamma x\left(1 - \frac{x}{k}\right)$. Therefore, we have the solution $W \leq \frac{\gamma x\left(1 - \frac{x}{k}\right)}{K_1}$ as t→ ∞.

This implies that all the solutions are bounded if the condition $0 \leq W \leq \frac{\gamma x\left(1 - \frac{x}{k}\right)}{K_1}$ holds.

Hence, all the solutions of the system are uniformly bounded in Q = {(x, y, z)$\in R_+^3 : 0 \leq W \leq \frac{\gamma x\left(1 - \frac{x}{k}\right)}{K_1} + \varepsilon$} for all ε is positive.

Equilibria:

Now, we desire to find the possible non negative equilibrium points for the system and also to analyze the stability criterion of these points. Let the possible equilibrium points for the system are given below:

i) the trivial equilibrium point $F_1^e(0,0,0)$
ii) the predator free equilibrium point $F_2^e(k,0,0)$
iii) the co-existing equilibrium point $F_3^e(x^*,y^*,z^*)$, where, $x^* = \frac{\{(\varepsilon_2-\mu)(\beta+\varepsilon_1)-\beta s\}k}{(\beta+\varepsilon_1)(km-\mu)}$,

$y^* = \frac{s\gamma\{(\beta+\varepsilon_1)(km-\varepsilon_2)+\beta s\}}{\alpha(\beta+\varepsilon_1)^2(km-\mu)}$, $z^* = \frac{\gamma(\beta+\varepsilon_1)(km-\varepsilon_2)+\beta s}{\alpha(\beta+\varepsilon_1)(km-\mu)}$.

The Jacobian matrix of the system (1) is given by

$$J(E) = \begin{bmatrix} \gamma-\frac{2x\gamma}{k}-\alpha z & 0 & mz-\frac{\mu z}{k} \\ 0 & -(\beta+\varepsilon_1) & \beta \\ -\alpha x & s & mx-\varepsilon_2+\mu(1-\frac{x}{k}) \end{bmatrix}$$

Theorem 3: The trivial equilibrium point $F_1^e(0,0,0)$ always unstable.

Proof: At the trivial equilibrium point $F_1^e(0,0,0)$ the Jacobian matrix becomes

$$J(F_1^e) = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & -(\beta+\varepsilon_1) & \beta \\ 0 & s & -\varepsilon_2+\mu \end{bmatrix}$$

Its characteristic equation is $(\gamma-\lambda)\{(\lambda^2+\beta+\varepsilon 1+\varepsilon 2-\mu\lambda+\beta+\varepsilon 1-\varepsilon 2+\mu-\beta s=0$. Here, $\lambda=\gamma$ is the positive root of the above system at this equilibrium point. So, the trivial equilibrium point $F_1^e(0,0,0)$ is always unstable.

Local stability and Global stability of the crisp model:

In this portion we explain the local stability and global stability of all possible equilibrium points of the crisp model (1) by the following theorem.

Theorem 4: If the condition $\varepsilon_2 > mk+\frac{\beta s}{\beta+\varepsilon_1}$ holds then the system (1) is locally asymptotically stable at predator free equilibrium point $F_2^e(k,0,0)$.

Proof: At the predator free equilibrium point $F_2^e(k,0,0)$ the Jacobian matrix for the system (1) is

$$J(F_2^e) = \begin{bmatrix} -\gamma & 0 & 0 \\ 0 & -\beta-\varepsilon_1 & \beta \\ -\alpha k & s & mk-\varepsilon_2 \end{bmatrix}$$

The characteristic equation is

$(\lambda+\gamma)[(\lambda^2-\lambda(mk-\varepsilon_2-\beta-\varepsilon_1)+\varepsilon_1\varepsilon_2-\beta mk+\beta\varepsilon 2-mk\varepsilon 1-\beta s=0.$

The characteristic roots are $\lambda_1 = -\gamma, \lambda_{2,3} = \frac{(mk-\varepsilon_2-\beta-\varepsilon_1)\pm\sqrt{(mk-\varepsilon_2-\beta-\varepsilon_1)^2-4(\varepsilon_1\varepsilon_2-\beta mk+\beta\varepsilon_2-mk\varepsilon_1-\beta s)}}{2}$.

Thus, if $\varepsilon_2 > mk+\frac{\beta s}{\beta+\varepsilon_1}$ then all the eigenvalues of the system at point $F_2^e(k,0,0)$ are either negative or have negative real parts. So, the system becomes locally asymptotically stable at F_2^e. Otherwise, the system will be unstable.

Theorem 5: if the conditions $\tau_1 = \frac{km-\mu}{\alpha k}\ \varepsilon_2-\mu-s\tau_2 \geq 0$ and $\tau_2\varepsilon_1+\beta(\tau_2-1)\geq 0$ holds then the system (1) are globally asymptotically stable at predator free equilibrium point.

Proof: Now, we construct a Lyapunov function

$$V(x,y,z)=\tau_1\int_k^x\frac{x-k}{x}dx+\tau_2\int_0^y dy+\int_0^z dz$$

Differentiating both sides with respect to t we get,

$$\frac{dV}{dt} = \tau_1\left(\frac{x-k}{x}\right)\frac{dx}{dt}+\tau_2\frac{dy}{dt}+\frac{dz}{dt}$$

$$= \tau_1\left(\frac{x-k}{x}\right)\left(\gamma x\left(1-\frac{x}{k}\right)-\alpha xz\right)+\tau_2(sz-\beta y-\varepsilon 1\, y+mxz+\beta y-\varepsilon 2\, z+\mu z(1-x/k)$$

$$=-\tau_1\frac{\gamma(x-k)^2}{k}+xz\left(m-\alpha\tau_1-\frac{\mu}{k}\right)-y(\beta\tau_2+\tau 2\varepsilon 1-\beta-z(\varepsilon 2-\mu-s\tau 2)$$

We choose $\tau_1 = \frac{km-\mu}{\alpha k}$, $\beta\tau_2+\tau_2\varepsilon_1-\beta\geq 0$, $\varepsilon_2-\mu-s\tau_2\geq 0$ then $\frac{dV}{dt}\leq 0$.

Therefore, the predator free equilibrium point is globally asymptotically stable.

Theorem 6: If all $S_i > 0, i = 1,2,3$ and $S_1S_2-S_3 > 0$ condition holds then the system (1) is locally asymptotical stable at the equilibrium point $F_3^e(x_1^*,y_1^*,z_1^*)$.

Proof: At the coexisting equilibrium point $F_3^e(x^*,y^*,z^*)$ the Jacobian matrix becomes

$$J(F_3^e) = \begin{bmatrix} \gamma-2\frac{\gamma x^*}{k}-\alpha z^* & 0 & mz^*-\frac{\mu z^*}{k} \\ 0 & -\beta-\varepsilon_1 & \beta \\ -\alpha x^* & s & mx^*-\varepsilon_2+\mu(1-\frac{x^*}{k}) \end{bmatrix}$$

The characteristic equation is

$$P(\lambda) = \lambda^3+S_1\lambda^2+S_2\lambda+S_3 = 0, \quad (6)$$

where, $S_1 = -(\gamma - 2\frac{\gamma x^*}{k} - \alpha z^* - \beta - \varepsilon_1 + mx^* - \varepsilon_2 + \mu(1 - \frac{x^*}{k}))$,

$S_2 = -\left\{\left(mx^* - \varepsilon_2 + \mu\left(1 - \frac{x^*}{k}\right)\right)(\beta + \varepsilon_1) + \beta s\right\} + \left(\gamma - 2\frac{\gamma x^*}{k} - \alpha z^*\right)\left(mx^* - \varepsilon_2 + \mu\left(1 - \frac{x^*}{k}\right) + \right.$
$\alpha x* mz*-\mu z*k-\gamma-2\gamma x*k-\alpha z*\beta+\varepsilon 1$,

$S_3 = -[\left(\gamma - 2\frac{\gamma x^*}{k} - \alpha z^*\right)\left\{(-\beta - \varepsilon_1)\left(mx^* - \varepsilon_2 + \right.\right.$
$\mu 1-x*k-\beta s\}+ mz*-\mu z*k-\alpha x*\beta+\varepsilon 1.$

Here, if all $S_i > 0, i = 1,2,3$ and $S_1S_2 - S_3 > 0$ then it satisfies all the conditions of Routh Hurwitz criterion. Hence, all of its roots have negative real parts and if coexisting equilibrium $F_3^e(x^*, y^*, z^*)$ exist then it is asymptotically stable.

Theorem 7: if the co-existing equilibrium point is feasible then it is globally asymptotically stable.

Proof: Taking a Lyapunov function
$\sigma(x, y, z) = \int_{x^*}^{x} \frac{x - x^*}{x} dx + P_1 \int_{y^*}^{y} \frac{y - y^*}{y} dy +$
$P2z*zz-z*zdz$

where, P_1 and P_2 are positive constants. It is to be determined in following steps.

Both side taking time derivatives of σ we have,

$$\frac{d\sigma}{dt} = \frac{x - x^*}{x}\frac{dx}{dt} + P_1\frac{y - y^*}{y}\frac{dy}{dt} + P_2\frac{z - z^*}{z}\frac{dz}{dt}$$

$=(x - x^*)[-\frac{\gamma}{k}(x - x^*) - \alpha(z - z^*)] - P_1(\beta + \varepsilon_1)(y - y^*)^2 + mP_2(x - x^*)(z - z^*) - P_2\varepsilon_2(z - z^*) - \frac{P_2\mu}{k}(x - x^*)(z - z^*) + (sP_1 + \beta P_2)(y - y^*)(z - z^*)$

$= -\frac{\gamma}{k}(x - x^*)^2 - \left(-mP_2 + \alpha + \frac{P_2\mu}{k}\right)(x - x^*)(z -$
$z*+ sP1+\beta P2y-y*z-z* -P2$
$\varepsilon_2(z - z^*) - P_1(\beta + \varepsilon_1)(y - y^*)^2$

If we choose $P_1 = -\frac{\beta k\alpha}{s(mk - \mu)}$ and $P_2 = \frac{k\alpha}{mk - \mu}$ $(s > 0, mk - \mu \neq 0)$ then we have $\frac{d\sigma}{dt} \leq 0$.

So, the coexisting equilibrium point $F_3^e(x^*, y^*, z^*)$ is globally asymptotically stable.

Hopf Bifurcation Analysis:

In this part we explain the possibility of Hopf bifurcation in the equilibrium point $F_3^e(x^*, y^*, z^*)$ with the help of parameter q. We choose $\lambda(q) = a(q) + ib(q)$ being the eigenvalue of the characteristic equation $\lambda^3 + F\lambda^2 + G\lambda + H = 0$ which is gained from the Jacobian $J(F_3^e)$. Putting the value of λ We have

$$a^3 - 3ab^2 + Fa^2 - Fb^2 + Ga + H = 0 \tag{7}$$

and

$$3a^2b - b^3 + 2Fab + Gb = 0 \tag{8}$$

We substitute $q = q_c$ such that $a(q_c) = 0$. We have purely imaginary solution for the characteristic equation (4). Now we put $a = 0$ in (7) and (8). Then we have

$$-Fb^2 + H = 0 \tag{9}$$

and

$$-b^3 + Gb = 0 \tag{10}$$

From (9) and (10) we get $b(q_c) = \sqrt{G(q_c)}$ and $F(q_c)G(q_c) - H(q_c) = 0$.

Therefore, eigenvalues of the equation (6) are $\lambda_1(q_c) = -F(q_c)$, $\lambda_2(q_c) = i\sqrt{G(q_c)}$, $\lambda_3(q_c) = -i\sqrt{G(q_c)}$ at the point $q = q_c$.

Theorem 8: The system (1) enters into Hopf bifurcation around interior equilibrium F_3^e, when the value of q passes its critical value q_c. At $q = q_c$ the system underlies Hopf bifurcation in the following condition

i) $F(q_c)G(q_c) - H(q_c) = 0$
ii) $F(q_c)\frac{dG(q_c)}{dq_c} + G(q_c)\frac{dF(q_c)}{dq_c} - \frac{dH(q_c)}{dq_c} \neq 0$ where $G(q_c) \neq 0$.

Proof: We assume that the formations of the roots of the characteristic equation (6) are $\lambda_1(q_c) = -F(q_c)$, $\lambda_2(q_c) = i\sqrt{G(q_c)}$, $\lambda_3(q_c) = -i\sqrt{G(q_c)}$ at the point $q = q_c$.

For all values of q, the root of the characteristic equation (6) becomes

$$\lambda_1(q) = a(q) + ib(q)$$

$$\lambda_2(q) = a(q) - ib(q)$$

$$\lambda_3(q) = -F(q)$$

Performing the transversality condition, we get the following at point $q = q_c$

$$\frac{d}{dr}\left(Re(\lambda(q))\right) \neq 0$$

Now, differentiating equation (6) and (7) with respect to q at the point $q = q_c$ and then we put $a = 0$, we have

$$A_1(q_c)\frac{da(q_c)}{dq_c} + A_2(q_c)\frac{db(q_c)}{dq_c} + A_3(q_c) = 0$$

$$B_1(q_c)\frac{da(q_c)}{dq_c} + B_2(q_c)\frac{db(q_c)}{dq_c} + B_3(q_c) = 0 \qquad (10)$$

where, $A_1(q_c) = -3b^2(q_c) + G(q_c), B_1(q_c) = 2b(q_c)F(q_c),\ A_2(q_c) = -2b(q_c)F(q_c),\ B_2(q_c) = -3b^2(q_c) + G(q_c),$
$A_3(q_c) = \frac{dH(q_c)}{dq_c} - b^2(q_c)\frac{dF(q_c)}{dq_c}, B_3(q_c) = b(q_c)\frac{dG(q_c)}{dq_c}$

Now solving the system (10), we have

At the point $q = q_c$,

$$\frac{d(Re(\lambda(q)))}{dq} = \frac{-2F(q_c)b^2(q_c)\frac{dG(q_c)}{dq_c} - \left(-3b^2(q_c)+G(q_c)\right)\left(\frac{dH(q_c)}{dq_c} - b^2(q_c)\frac{dF(q_c)}{dq_c}\right)}{(-3b^2(q_c)+G(q_c))^2+(2b(q_c)F(q_c))^2}$$

We put the value of b and the above expressing shows that $\frac{d}{dq}\left(Re(\lambda(q))\right) \neq 0$, at the point $q = q_c$

If $F(q_c)\frac{dG(q_c)}{dq_c} + G(q_c)\frac{dF(q_c)}{dq_c} - \frac{dH(q_c)}{dq_c} \neq 0$ and $G(q_c) \neq 0$

Hence the transversality condition holds and this implies that Hopf bifurcation occurs at $q = q_c$.

Complete the proof.

6. Numerical Results:

The global dynamics of system (1) is investigated numerically using MATLAB *ode45* codes and taking the input data of the parameters from Table 2. To study the feasibility of the fuzzy model of the concerned prey predator model, all biological parameters are hypothesized to be imprecise in nature and these are considered as triangular fuzzy numbers. Using MATCONT we indicate the Hopf point, branch point and the limit cycle in bifurcation scenario. The following numerical examples are considered to discuss the theoretical study.

Result-1 Crisp Model In this section we have applied the following parameters which are used above system. We have got the interior equilibrium point F_3^e(1.37, 0.7455, 1.5674) by the value of crisp model involved parameters which are taken from Table 2. **Fig. 2** has been drawn for the equilibrium points. The equilibrium is locally asymptotically stable. We have represented in **Fig. 3** for phase space diagram in both crisp environment and fuzzy environment. We have studied the bifurcation analysis of the system (1) for the parameters s and β. The existence of the periodic orbit bifurcating from equilibrium point of the system (1) is examined. We have shown the continuation curves of the equilibrium with the variation of parameter s in **Fig. 4.** From **Fig. 4** it is seen that Hopf point is in the positive quadrant. The Hopf point (H) is situated in $(x, y, z, s) = (0, 2.40287, 1.6315796, 0.03818)$ and correspond-ing eigenvalues are $(-4.91, \pm i0.000573208)$. We have the first Lyapunov coefficient 0.024924. This gives subcritical Hopf bifurcation which represent unstable limit cycle and bifurcation from equilibrium in there. Also, we have the branch point $(x, y, z, s) = (0, 2.402871, 1.631572, 6.038182)$. Ag-ain, we have represented the continuation curves of the equilibrium with the variation of parameter β in **Fig. 5.** We have the Hopf point (H) which is located in $(x, y, z, \beta) = (0, 0.523944, 1.631579, 2.522364)$ and corresponding eigenvalues are $(-6.88236, \pm i0.00109632)$. The first Lyapunov coefficient is 0.05713841. This represents subcritical Hopf bifurcation. We have observed branch point $(x, y, z, \beta) = (0, 0.523944, 1.631579,\ 2.522368)$ in the **Fig. 5.**

Result-2 Fuzzy model

Since our environment is constantly changed by the natural disaster, human activities etc. The biological parameters are fluctuated due to it. So, we have considered that all the model involved parameters are taken as triangular fuzzy numbers. We have shown crisp value and corresponding fuzzy value of the parameters in the Table 2. We have drawn the solution curve for the fuzzy value in the **Fig. 2(b).** From figure it is observed that mature predator population predate both the population prey and immature predator in the environment. So, the mature predator will be stronger than other population. The system (2) has equilibrium point around F_3^e(1.3763, 3.2182, 1.567). **Fig. 3 (b)** represents the phase space diagram for different values of initial value in the fuzzy environment.

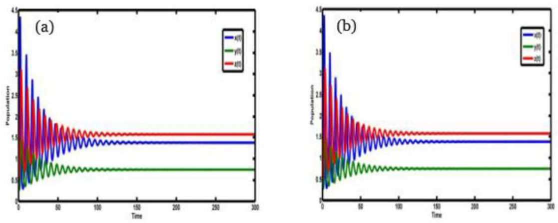

Figure 2: profile of population of model system (1) around interior equilibrium point in the (a) crisp environment (b) fuzzy environment.

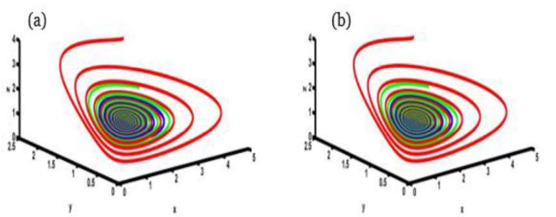

Figure 3: Phase space diagram for the model system (1) in the (a) crisp environment (b) fuzzy environment

Result-3 Defuzzified model

Firstly, we have formed crisp model then we have transformed to the fuzzy model. Again, fuzzy model has been converted to the defuzzified model by using the GMIV formula. The defuzzified values of the model involved parameters have been shown in the Table 3 for different optimism w. From **Fig. 6** it is observed that all the population has significantly changed in population. So, from these figures we have concluded that all the species in the environment has been influenced by the changing parameters.

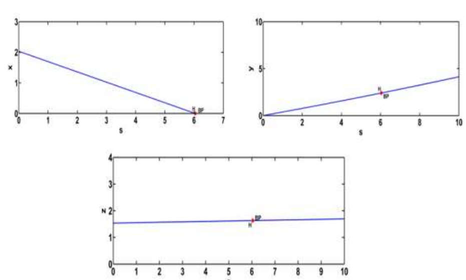

Figure 4: Continuation curves of the equilibrium with the variation of reproduction rate of mature predator s (H represents Hopf point and BP represents branch point)

Result- 4 Stochastic Model

The solution of the stochastic differential equation model can be determined by the Euler-Maruyama method. We have used the Euler-Maruyama method for the system (4) and we have obtained

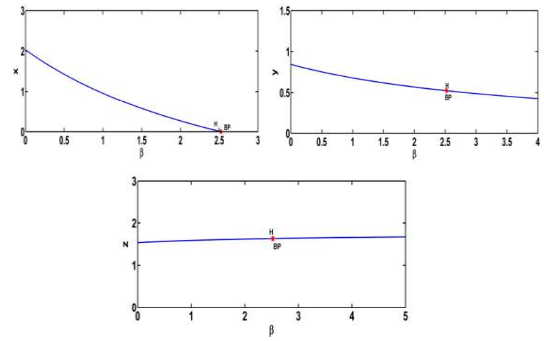

Figure 5: Continuation curve of the equilibrium with the variation of translation rate from immature predator to mature predator β. (H represent Hopf point and BP represent branch point)

Crisp parameters	value	Fuzzy parameters	value
γ	1.55	$\tilde{\gamma}$	(1.395, 1.55, 1.705)
k	35	$\tilde{k}$	(31.5, 35, 38.5)
α	0.95	$\tilde{\alpha}$	(0.855, 0.95, 1.045)
s	1.95	$\tilde{s}$	(1.755, 1.95, 2.145)
β	0.55	$\tilde{\beta}$	(0.495, 0.55, 0.605)
ε_1	3.55	$\tilde{\varepsilon}_1$	(3.195, 3.55, 3.905)
m	0.4	$\tilde{m}$	(0.36, 0.4, 0.44)
ε_2	0.85	$\tilde{\varepsilon}_2$	(0.765, 0.85, 0.935)
μ	0.04	$\tilde{\mu}$	(0.036, 0.04, 0.044)

Table 2: Values of the parameters

$$x(i+1) = x(i) + (x(i)\left(\gamma\left(1 - \frac{x(i)}{k}\right) - \alpha z(i)\right)h + \theta_1 x(i)\sqrt{h}\,N(0,1);$$

$$y(i+1) = y(i) + (sz(i) - \beta y(i) - \varepsilon_1 y(i))h + \theta_2 y(i)\sqrt{h}\,N(0,1);$$

$$z(i+1) = z(i) + z(i)(mx(i) - \varepsilon_2 + \mu(1 - x(i)/k) + \beta y(i))h + \theta_3 z(i)\sqrt{h}\,N(0,1);$$

for $i = 1,2,\ldots n$ and $x(0) = 0.8$, $y(0) = 0.6$, $z(0) = 0.4$, $\gamma = 1.55$, $k = 35$, $\alpha = 0.95$, $s = 1.95$, $\beta = 0.55$, $\varepsilon_1 = 3.55$, $m = 0.4$, $\varepsilon_2 = 0.85$, $\mu = 0.04$. The number of steps in the mesh $N = 50000$, $T = 10$, $h = T/N$. The values of the parameters $\theta_1, \theta_2, \theta_3$ are taken hypothetically. We have studied above for the stochastic prey predator system (a) the phase portrait of three species prey (x), immature predator (y), mature predator (z) in different noise and (b) the evolution in time of the species x, y, z. In **Fig. 7** we have shown the phase portrait for the stochastic prey predator system. The **Fig. 7** and **Fig. 8** represents the limit cycle on equilibrium point. If the limit cycle is unstable then the population becomes extinct. From **Fig. 8** we have observed the significant fluctuation of the population for the different noise. From the **Fig. 8(a)** and **Fig. 8(b)** It is seen that all the population $x(t), y(t), z(t)$ are extinct after few days. From **Fig. 8(c)** we have seen all the population $x(t), y(t), z(t)$

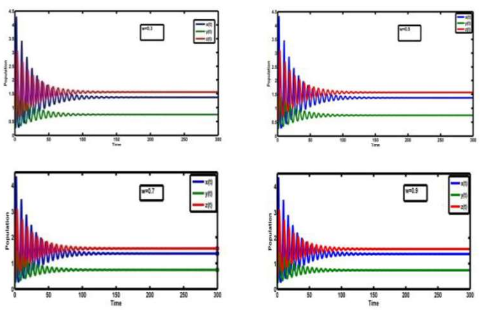

Figure 6: Profile of population for the different degree of optimism w

Parameters	Fuzzy value	Defuzzified value $w = 0.3$	$w = 0.5$	$w = 0.7$	$w = 0.9$
γ	(1.395, 1.55, 1.705)	1.52	1.55	1.57	1.59
k	(31.5, 35, 38.5)	34.5	35	35.46	35.93
α	(0.855, 0.95, 1.045)	0.937	0.95	0.962	0.975
s	(1.755, 1.95, 2.145)	1.924	1.95	1.976	2.002
β	(0.495, 0.55, 0.605)	0.542	0.55	0.557	0.564
ε_1	(3.195, 3.55, 3.905)	3.50	3.55	3.59	3.644
m	(0.36, 0.4, 0.44)	0.394	0.4	0.405	0.410
ε_2	(0.765, 0.85, 0.936)	0.838	0.85	0.861	0.872
μ	(0.036, 0.04, 0.044)	0.0394	0.04	0.0405	0.0410

Table 3: Defuzzified value of the parameters

are permanent for different noise $\theta_1 = 1.7, \theta_2 = 1.5, \theta_3 = 1.8$. In Fig. 9 indicates the evolution in time of the predator and prey with increasing sequence noise. From Fig. 9(a) we have observed that $x(t), z(t)$ are permanent and $y(t)$ is extinct for the noise $\theta_1 = \theta_2 = \theta_3 = 1.2$, from Fig. 9(b) we have got $x(t)$ is extinct but $y(t), z(t)$ are permanent for the noise $\theta_1 = \theta_2 = \theta_3 = 1.7$ and from Fig. 9(c) we have $y(t), z(t)$ are extinct for the noise $\theta_1 = 1.7, \theta_2 = 1.5, \theta_3 = 1.8$. When the noise $\theta_1 = \theta_2 = \theta_3 = 1.2$, all the curve follows closely the deterministic curve. When $\theta_1 = \theta_2 = \theta_3 = 1.7$ the system becomes oscillates with presence of the noise. Also, the system exhibits oscillation for the

different noise $\theta_1 = 1.7, \theta_2 = 1.5, \theta_3 = 1.8$.

Sensitivity Analysis of the parameters:

Sensitivity of *k*

Here we have discussed sensitivity of the carrying capacity k. (i) For the large carrying capacity $k = 350$ (**Fig. 10(a)**) we have observed that all the curve becomes oscillation. It is concluded that all the population exist in the environment. (ii) For the moderate value of carrying capacity $k = 35$(**Fig. 2(a)**) we have determined that the system is locally asymptotically stable around its equilibrium point.

(iii) When carrying capacity $k = 0.35$ (**Fig. 10(b)**), then we conclude only prey population is exist and other population becomes abolished in the environment.

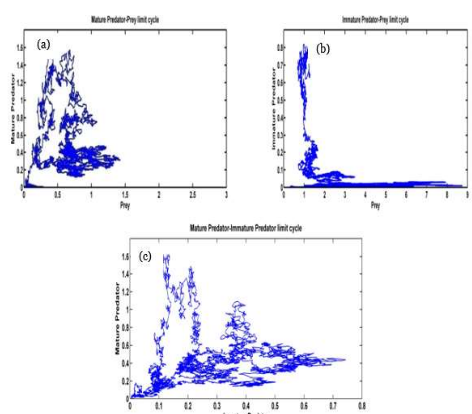

Figure 7: The phase portraits of (a) mature predator with respect to prey (b) immature predator with respect to prey with noise $\theta_1 = \theta_2 = \theta_3 = 2.0$ (c) mature predator with respect to immature predator with different noise $\theta_1 = 1.7, \theta_2 = 1.5, \theta_3 = 1.8$.

Sensitivity of parameter *s*

In this portion we have examined the sensitivity of the parameter s. When $s \geq 6.5$, the solution curves of the system (1) do not exist. (i) For large value of the reproduction rate of mature predator s (**Fig. 11(a)**) the prey population vanishes in the environment. (ii) For the moderate value of s (**Fig. 2(a)**) all the species are coexisted in the nature. (iii) It is evident from the **Fig. 11(b)** prey population and mature predator population are coexist in the environment.

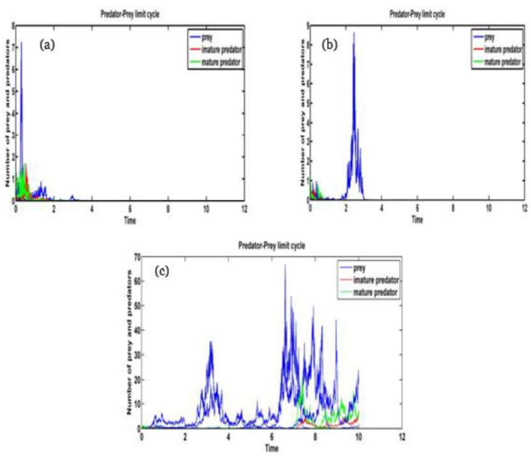

Figure 8: The evolution in time of the predator and of the prey with noise (a) $\theta_1 = \theta_2 = \theta_3 = 3.0$, (b) $\theta_1 = \theta_2 = \theta_3 = 3.5$ and (c) $\theta_1 = 1.7, \theta_2 = 1.5, \theta_3 = 1.8$

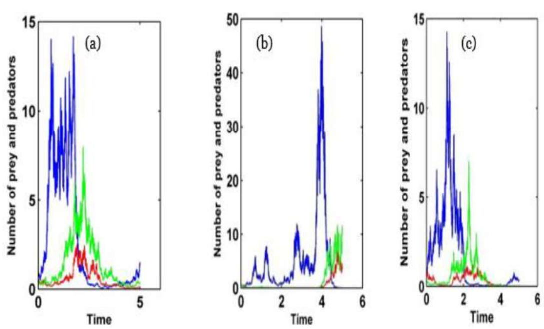

Figure 9: The evolution in time of the predator and of the prey with increasing sequence noise (a) $\theta_1 = \theta_2 = \theta_3 = 1.2$ and (b) $\theta_1 = \theta_2 = \theta_3 = 1.7$ (c) $\theta_1 = 1.7, \theta_2 = 1.5, \theta_3 = 1.8$ (blue line represent x(t), red line represents y(t), green line represents z(t))

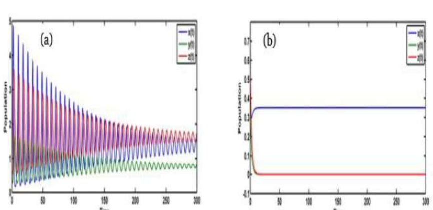

Figure 10: solution curves for (a) very large carrying capacity $k = 350$ and (b) low carrying capacity $k = 0.35$.

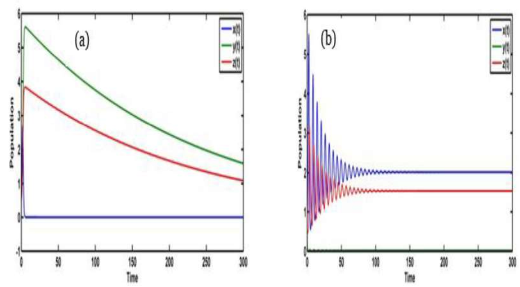

Figure 11: Solution curves (a) for large value of $s = 6.0$ and (b) for low value of $s = 0.05$

Sensitivity of the parameter μ

In here we have investigated that the solution curve does not exist for maximum growth rate for mature predator $\mu \geq 0.6$. For moderate value of $\mu = 0.04$ (**Fig. 2 (a)**) and low value of $\mu = 0.005$ (**Fig. 12 (b)**) we have seen that very slight deflection arises between prey and mature predator population. But immature predator remains unchanged.

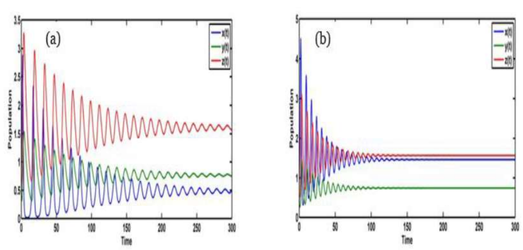

Figure 12: Profile of population for (a) $\mu = 0.4$ and (b) $\mu = 0.005$

7. Conclusion

The model formulation section is divided into four subsections, they are namely crisp model, fuzzy model, defuzzified model and stochastic model. We take all biological parameters as triangular fuzzy numbers (TFNs) in the fuzzy model. With the help of graded mean integration technique, we convert the fuzzy prey predator model to defuzzified model. We illustrate the crisp model as stochastic model with the help of Markov chain process. Persistence in mean and extinction of the system are examined in the stochastic prey predator model subsection. The positivity, boundedness, equilibrium point, local asymptotically stable, global asymptotically stable of the interior equilibrium point of the model, Hopf bifurcation analysis are discussed. We describe the crisp model example, fuzzy model example, defuzzified model example and stochastic model example in the numerical results portion. Solution curves around interior equilibrium point, phase space diagram of the model (1) is drawn in both (crisp, fuzzy) environments. Solution curves for the different optimism w are drawn. Slight fluctuation in the figure is observed. The existence of Hopf point (**H**) and branch point (**BP**), limit cycle is looked into. In our analytical results we have found that the white noise perturbation plays an important role in extinction as well as persistence of prey and predator populations. We perturb the model with respect to white noise around the intrinsic growth rate of prey and death rate of predator populations. Graphically we have shown the limit cycle of mature predator with respect to prey, immature predator with respect to prey with the noise $\theta_1 = \theta_2 = \theta_3 = 2.0$ and mature predator with respect to immature predator with different noise $\theta_1 = 1.7, \theta_2 = 1.5, \theta_3 = 1.8$. also we represent The evolution in time of the predator and of the prey with increasing sequence noise $\theta_1 = \theta_2 = \theta_3 = 1.2$, $\theta_1 = \theta_2 = \theta_3 = 1.7$ and $\theta_1 = 1.7, \theta_2 = 1.5, \theta_3 = 1.8$ (blue line represent $x(t)$, red line represent $y(t)$, green line represent $z(t)$). At last, we describe the sensitivities of the parameters k, μ, s with taking their large, moderate and low value. To improve the research area of the mathematical biology we may consider this type of fuzzification, defuzzification method, stochasticity in different models. There are some investigations for stochastic ecological models. It is cabbalistic for realistic world. It is more eagerness to study evolutionary dynamics of stochastic evolutionary model. So, we decide these for future work.

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