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μ-almost Ricci solitons on hyperbolic Kenmotsu manifolds

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Abstract
The aim of the current article is to study and investigate μ-almost Ricci solitons on hyperbolic Kenmotsu manifolds. In this manner, we prove the second order Ricci tensor field S is parallel along the timelike vector field ζ if and only if $\zeta(\mu) = 0$. It is also shown that the soliton is shrinking and the potential vector field V is point-wise collinear with constant multiple of the timelike vector field ζ. Further, we obtain some results for μ-almost Ricci solitons on hyperbolic Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor.

Keywords: almost Ricci solitons, μ-almost Ricci solitons, hyperbolic Kenmotsu manifold, Codazzi type Ricci tensor, cyclic parallel Ricci tensor.

1. Introduction

A lot of progress has been made recently in the study of self-similar solutions of the Ricci flow.
In 1982, Hamilton [10] revealed the concept of Ricci flow to search out a canonical metric on a smooth manifold. A Ricci soliton is nothing but a natural generalized case of an Einstein metric and is defined on a Riemannian manifold (M, g) [5].
The Ricci flow is an evolution equation for metrics on a Riemannian manifold (M, g) defined as follows:

$$\frac{\partial}{\partial t}(g(t)) = -2S \tag{1.1}$$

where S is the Ricci tensor of type $(0,2)$ in a Riemannian manifold (M, g).

A Ricci soliton we mean a Riemannian manifold (M, g) together with a smooth vector field V (called the potential vector field) and a real scalar λ satisfying

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0 \tag{1.2}$$

where $\mathcal{L}_V$ indicates Lie-derivative operator of g along the smooth vector field V on M [3].

The Ricci soliton is said to be shrinking, steady and expanding if λ is negative, zero and positive respectively. If the potential vector field V of the Ricci soliton (g, V, λ) is Killing, then the soliton is trivial (Einstein) provided the dimension of M is greater than 2. It is clear that if V is conformal then, the manifold is Einstein. Ricci solitons have been investigated by several geometers such as ([1], [6], [10], [15]). The so called Ricci soliton becomes the almost Ricci soliton if $\lambda : M \to R$ is a C^∞ function (called the soliton function).

In [13], Pigola et al. introduced the above notion clearly which is a generalization of Ricci soliton. Recently, the idea of almost Ricci soliton was generalized by Gomes et al. [8] and named as μ-almost Ricci soliton. An μ-almost Ricci soliton is a complete Riemannian manifold (M, g) with a smooth vector field V on M, a soliton function $\lambda : M \to R$ and a smooth function $\mu : M \to R$ satisfying the equation

$$\frac{\mu}{2}\mathcal{L}_V g + S + \lambda g = 0 \tag{1.3}$$

We denote the μ-almost Ricci soliton by (g, V, μ, λ). If the potential vector field V is gradient of some C^∞ function on M, then the previous equation reduces to gradient μ-almost Ricci soliton.
Furthermore, if the soliton function λ is constant, 1-almost Ricci soliton is a Ricci soliton. More recently, Ghosh and Patra in [7] has studied μ-almost Ricci solitons within the context of K-contact metric manifolds. They got a few fascinating results.

A Gray [9] revealed the idea of cyclic parallel Ricci tensor and Ricci tensor of Codazzi type. Codazzi type of Ricci tensor means that the Levi-Civita connection ∇ of such metric is a Yang-Mills connection while keeping the metric of the

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manifold fixed. A Riemannian manifold (M, g) is said to have cyclic parallel Ricci tensor if its Ricci tensor field S of type $(0,2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \quad (1.4)$$

for any $X, Y, Z \in \chi(M)$

Again, a Riemannian manifold (M, g) is said to have Ricci tensor of Codazzi type if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z) \quad (1.5)$$

for any $X, Y, Z \in \chi(M)$. Ki et al. [11] proved that Carten hypersurfaces are manifold with non-parallel Ricci tensor satisfies cyclic parallel Ricci tensor while Bourguigon [4] proved that the interesting result that any metric with Codazzi type of Ricci tensor on a compact orientable 4-manifold with non-vanishing signature is Einstein.

The organization of the current article is as follows:

After introduction, Section 2 contains preliminaries on hyperbolic Kenmotsu manifolds. In section 3, we consider an μ-almost Ricci solitons on hyperbolic Kenmotsu manifolds. Here, prove that the Ricci tensor is parallel along the timelike vector field ζ if and only if $\zeta(\mu) = 0$. In this section we have also shown that the soliton is shrinking in a hyperbolic Kenmotsu manifold when a soliton vector field V is point-wise collinear with the timelike vector field ζ. In section 4, we study the existence of μ-almost Ricci solitons on hyperbolic Kenmotsu manifolds admitting Codazzi type of Ricci tensor field. In section 5, we obtain a results for μ-almost Ricci solitons on hyperbolic Kenmotsu manifolds admitting. cyclic parallel Ricci tensor field.

1. Preliminaries

An odd dimensional smooth manifold M^{2n+1} is named to be an almost hyperbolic contact metric manifold if it admits a timelike vector field ζ, a 1-form η, a fundamental tensor field φ of type $(1,1)$ and a semi-Riemannian metric g satisfying [14]:

$$\varphi^2(X) = X + \eta(X)\zeta \quad (2.1)$$
$$\eta(\zeta) = -1 \Rightarrow \varphi(\zeta) = 0 \quad (2.2)$$
$$rank\ \varphi = 2n \quad (2.3)$$
$$\eta \circ \varphi = 0 \quad (2.4)$$
$$g(\varphi X, \varphi Y) = -g(X,Y) - \eta(X)\eta(Y) \quad (2.5)$$
$$g(\varphi X, Y) + g(X, \varphi Y) = 0 \quad (2.6)$$
$$g(X, \zeta) = \eta(X) \quad (2.7)$$

for all $X, Y \in \chi(M^{2n+1})$. Then the structure (φ, ξ, η, g) on manifold M^{2n+1} named as almost hyperbolic contact metric structure.

If an almost hyperbolic contact metric manifold M^{2n+1} is fulfilled the following condition:

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\zeta - \eta(Y)\varphi X \quad (2.8)$$

then, the manifold M^{2n+1} is called a hyperbolic Kenmotsu manifold [2]. where ∇ denotes the Levi-Civita connection of the metric g.

From the antecedent equation it is clear that

$$\nabla_X \zeta = -X - \eta(X)\zeta \quad (2.9)$$

and

$$(\nabla_X \eta)Y = -g(X,Y) - \eta(X)\eta(Y) \quad (2.10)$$

In a $(2n + 1)$-dimensional hyperbolic Kenmotsu space form [12], we have

$$R(X,Y)\zeta = \eta(Y)X - \eta(X)Y \quad (2.11)$$

$$R(X,\zeta)\zeta = -X - \eta(X)\zeta \quad (2.12)$$

$$R(\zeta,X)Y = g(X,Y)\zeta - \eta(Y)X \quad (2.13)$$

$$S(X,\zeta) = 2n\eta(X) \quad (2.14)$$

$$S(\zeta,\zeta) = -2n \quad (2.15)$$

$$Q\zeta = 2n\zeta \quad (2.16)$$

where R, S and Q are the curvature tensor, the Ricci tensor and the Ricci operator of M^{2n+1}, respectively.

In a hyperbolic Kenmotsu manifold we obtain

$$(\mathcal{L}_\zeta g)(X,Y) = -2\{g(X,Y) + \eta(X)\eta(Y)\} \quad (2.17)$$

for any $X, Y \in \chi(M^{2n+1})$.

3. μ-almost Ricci solitons on hyperbolic Kenmotsu manifolds

Let a hyperbolic Kenmotsu manifold M^{2n+1} admits an μ-almost Ricci soliton (g, ζ, μ, λ).
Then we have

$$\frac{\mu}{2}(\mathcal{L}_{\zeta} g)(X, Y) + S(X, Y) + \lambda g(X, Y) = 0 \qquad (3.1)$$

for any $X, Y \in \chi(M^{2n+1})$.
Now, using (2.17) in equation (3.1) gives

$$S(X, Y) = (\mu - \lambda)g(X, Y) + \mu\eta(X)\eta(Y) \qquad (3.2)$$

Putting $Y = \zeta$ in (3.2) and using (2.2), (2.14), we get

$$(\lambda + 2n) = 0 \qquad (3.3)$$

Since $\eta(X) \neq 0$, we get

$$\lambda = -2n \qquad (3.4)$$

Hence, from (3.2) and (3.4) it follows that

$$S(X, Y) = (\mu + 2n)g(X, Y) + \mu\eta(X)\eta(Y) \qquad (3.5)$$

for any $X, Y \in \chi(M^{2n+1})$.
Thus, we can state the following theorem:

Theorem 3.1. *If a $(2n + 1)$-dimensional hyperbolic Kenmotsu manifold M^{2n+1} admits an μ-almost Ricci soliton (g, ζ, μ, λ), then the soliton is shrinking with $= -2$.*

In view of the equation (3.5), we can state the following:

Corollary . . *If a (2 + 1)-dimensional hyperbolic Kenmotsu manifold $^{2+1}$ admits an -almost Ricci soliton (, , ,), then the manifold becomes an -Einstein.*

Next we prove the following theorem:

Theorem . . *If a (2 + 1)-dimensional hyperbolic Kenmotsu manifold $^{2+1}$ admits an -almost Ricci soliton (, , ,), then the Ricci tensor field of type (0,2) is parallel along the timelike vector field if and only if () = 0.*

Proof: Taking covariant differentiation of (3.5) with respect to we get

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{()() + ()()} (3.6)

In the view of the equation (2.10) we have

()(,) = (){()() + (,)} −
{()(,) + ()(,) + 2()()()}
(3.7)

Setting = in the above equation (3.7), we get

$$(\nabla_{\zeta} S)(X, Y) = \zeta(\mu)\{\eta(X)\eta(Y) + g(X, Y)\} \qquad (3.8)$$

Suppose the Ricci tensor field S of type (0,2) is parallel along ζ, then from (3.8) we have

$$\zeta(\mu)\{\eta(X)\eta(Y) + g(X, Y)\} = 0 \qquad (3.9)$$

for any $X, Y \in \chi(M^{2n+1})$.
It follows that $\zeta(\mu) = 0$.

Conversely, we assume that $\zeta(\mu) = 0$

Then from (3.8), it follows that $(\nabla_{\zeta} S)(X, Y) = 0$ for any $X, Y \in \chi(M^{2n+1})$. This shows that the Ricci tensor field S of type (0,2) is parallel along ζ. This finishes the proof.

Next we prove the following theorem:

Theorem 3.4. *If a $(2n + 1)$-dimensional hyperbolic Kenmotsu manifold M^{2n+1} admits an μ-almost Ricci soliton (g, ζ, μ, λ) and the potential vector field V is point-wise collinear with the timelike vector field ζ, then the vector field V is a constant multiple of ζ. Furthermore, the soliton is shrinking with $\lambda = -2n$.*

Proof: Let us consider the soliton vector field V is point-wise collinear with the timelike vector field ζ, then there exists a smooth function h such that $V = h\zeta$.
Then from (1.3) we derive

$$\mu\{g(\nabla_X h\zeta, Y) + g(X, \nabla_Y h\zeta)\} + 2S(X, Y) + 2\lambda g(X, Y) = 0 \qquad (3.10)$$

Utilizing the equation (2.9) in above equation (3.10), we get

$$\mu(\mathcal{L}_{\zeta} g)(X, Y) + \mu\{X(h)\eta(Y) + Y(h)\eta(X)\} + 2S(X, Y) + 2\lambda g(X, Y) = 0 \qquad (3.10)$$

With the help of the equation (2.17), we get

$$-2\mu h\{g(X,Y)+\eta(X)\eta(Y)\}+\mu X(h)\eta(Y)+$$
$$\mu Y(h)\eta(X)+2S(X,Y)+2\lambda g(X,Y)=0 \quad (3.11)$$

Replacing Y by ζ in the above equation (3.11) and using the equations (2.2), (2.7) and (2.14) yields

$$-\mu X(h)+\{\mu\zeta(h)+4n+2\lambda\}\eta(X)=0 \quad (3.12)$$

Again replacing X by ζ in the above equation (3.12) and using (2.7) we get

$$\mu\zeta(h)=-2n-\lambda \quad (3.13)$$

Substituting the value of $\mu\zeta(h)$ in the equation (3.12) we get

$$d(h)=\frac{(2n+\lambda)}{\mu}\eta \quad (3.14)$$

Applying exterior derivative on both sides of the equation (3.14) and using the Poincare lemma $d^2\equiv 0$, we get

$$\frac{(2n+\lambda)}{\mu}d\eta+d\left\{\frac{2n+\lambda}{\mu}\right\}\eta=0 \quad (3.15)$$

Taking wedge product of the above equation (3.15) with η , we have

$$\left\{\frac{2n+\lambda}{\mu}\right\}\eta\wedge d\eta=0 \quad (3.16)$$

Since $\eta\wedge d\eta\neq 0$, in a hyperbolic Kenmotsu manifold, we infer

$$\lambda=-2n \quad (3.17)$$

Now substituting the value of λ in the equation (3.14) we have $d(h)=0\Rightarrow h=$ constant on M^{2n+1}.

Consequently, the equation (3.11) reduces to

$$S(X,Y)=(\mu h+2n)g(X,Y)+\mu h\eta(X)\eta(Y) \quad (3.18)$$

Comparing it with the equation (3.5), we get $h=1$. Again from the equation (3.17), $\lambda=-2n>0$ for $n>0$. Thus, the soliton is shrinking.

This finishes the proof.

4. μ-almost Ricci solitons on hyperbolic Kenmotsu manifolds with Codazzi type of Ricci tensor field

First, we suppose that the hyperbolic Kenmotsu manifolds M^{2n+1} admits an μ-almost Ricci solitons satisfy the condition (1.5), i.e., the Ricci tensor field S is of Codazzi type.

Then from (1.5) we have

$$(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)=0 \quad (4.1)$$

for any $X,Y\in\chi(M^{2n+1})$.

Therefore, taking covariant differentiation of (3.5) with respect to Z we obtain

$$(\nabla_Z S)(X,Y)=Z(\mu)\{\eta(X)\eta(Y)+g(X,Y)\}+$$
$$\mu\{\eta(X)(\nabla_Z\eta)Y+\eta(Y)(\nabla_Z\eta)X\} \quad (4.2)$$

In the view of the equation (2.10) we infer

$$(\nabla_Z S)(X,Y)=Z(\mu)\{\eta(X)\eta(Y)+g(X,Y)\}-$$
$$\mu\{\eta(X)g(Z,Y)+\eta(Y)g(Z,X)+2\eta(X)\eta(Y)\eta(Z)\} \quad (4.3)$$

Utilizing (4.3) in the equation (4.1) entails that

$$X(\mu)\{\eta(Z)\eta(Y)+g(Z,Y)\}-Y(\mu)\{\eta(X)\eta(Z)+$$
$$g(X,Z)\}-\mu\{\eta(Y)g(X,Z)+\eta(X)g(Y,Z)\}=0 \quad (4.4)$$

Putting $Z=\zeta$ in the above equation (4.4), we lead

$$\mu\eta(X)\eta(Y)=0 \quad (4.5)$$

for any $X,Y\in\chi(M^{2n+1})$.

It follows that $\mu=0$. Hence a hyperbolic Kenmotsu manifold with Codazzi type of Ricci tensor does not admit a proper μ-almost Ricci soliton. Thus we conclude the following:

Theorem 4.1. *A $(2n+1)$-dimensional hyperbolic Kenmotsu manifold M^{2n+1} with Codazzi type of Ricci tensor field does not admit a proper μ-almost Ricci soliton (g,ζ,μ,λ).*

Now for $\mu=0$, the equation (3.5) reduces to

$$S(X,Y)=2ng(X,Y) \quad (4.6)$$

for any $X,Y\in\chi(M^{2n+1})$.

Hence, the manifold reduces to an Einstein manifold. Thus we state the following:

Corollary 4.2. *If a $(2n+1)$-dimensional hyperbolic Kenmotsu manifold M^{2n+1} with Codazzi type of Ricci tensor field admits an μ-almost Ricci soliton (g, ζ, μ, λ), then the manifold becomes an Einstein manifold.*

5. μ-almost Ricci solitons on hyperbolic Kenmotsu manifolds with cyclic parallel Ricci tensor field

Here, we suppose that the hyperbolic Kenmotsu manifolds M^{2n+1} admits an μ-almost Ricci solitons satisfying the condition (1.4), i.e., the Ricci tensor field S is cyclic parallel.
Then from (1.4) we have

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \quad (5.1)$$

for any $X, Y, Z \in \chi(M)$.
Utilizing (4.3) in the above equation (5.1), and setting $Z = \zeta$ and then maintaining the same procedure as in the proof of Theorem 4.1, we can easily obtain

$$\{\zeta(\mu) + 2\mu\}\{\eta(X)\eta\big(Y + g(X,Y)\big) = 0 \quad (5.2)$$

for any $X, Y \in \chi(M^{2n+1})$.
It follows that $\zeta(\mu) = -2\mu \Rightarrow D\mu = 2\mu\zeta$. Hence the gradient of the smooth function μ is point-wise collinear with the timelike vector field ζ. Thus we conclude the following:

Theorem 5.1. *If a $(2n+1)$-dimensional hyperbolic Kenmotsu manifold M^{2n+1} with parallel Ricci tensor field admits an μ-almost Ricci soliton (g, ζ, μ, λ), then the gradient of the smooth function μ is point-wise collinear with the timelike vector field ζ.*

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